Quantized Rational Chip-Firing

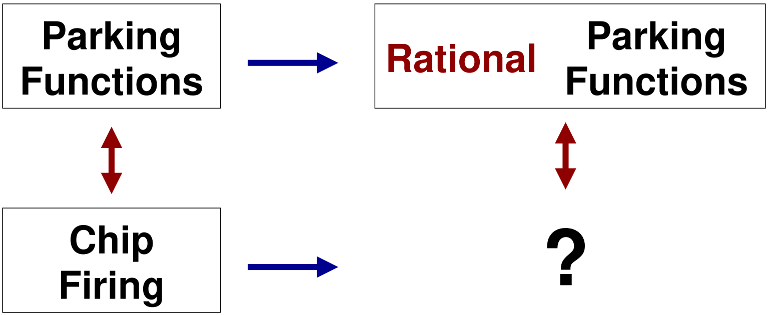
Spencer Backman (Univ. Vermont)
Nick Loehr (Virginia Tech)
Greg Warrington (Univ. Vermont)

Lattice Path Conference 2026
TU Wien, July 21

Goal

Generalize chip-firing to the “rational” case.

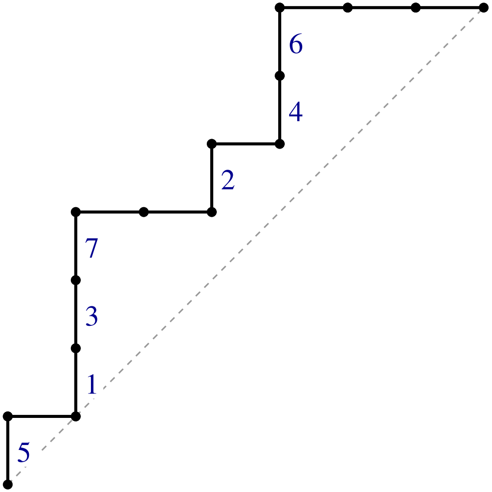
Goal


Parking Functions
Rational Parking Functions
Chip Firing
?

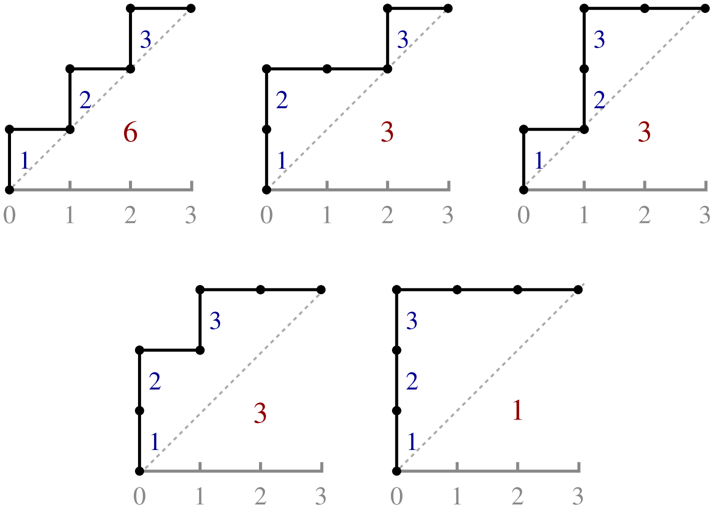
Parking functions

Parking Functions

PF: An order-7 example



PF: All 16 of order-3



PF: Connections

- (Pyke '59; Konheim & Weiss '66):

$$|PF_n| = (n+1)^{n-1}$$

- (Conj: HHLRU '05; Carlsson & Mellit '18):

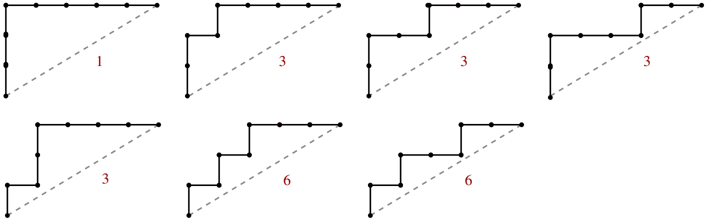
$$\mathrm{Frob}_{DR_n} = \nabla e_n = \sum_{P \in PF_n} q^{\mathsf{area}(P)} t^{\mathsf{dinv}(P)} F_{\mathrm{ides}(P)}.$$

- (Pak & Stanley '95): PF_n indexes regions of Shi arrangement.

- (Stanley '96): PF_n indexes the maximal chains of the poset of non-crossing partitions.

Parking Functions $\longrightarrow$ **Rational Parking Functions**

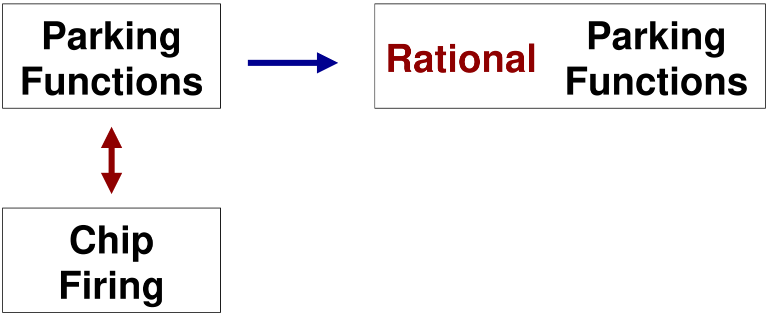
Consider only paths above $ay = bx$:



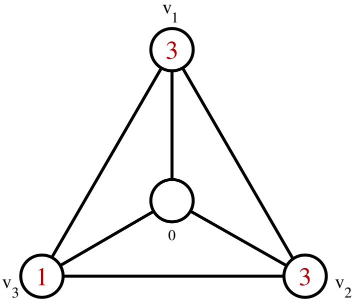
Fact: If $\gcd(a,b) = 1$,

- (Bizley '54): $\frac{1}{a+b}\binom{a+b}{a,b}$ rational Dyck paths and
- (Armstrong, Loehr & W '16): There are a^{b-1} rational parking functions.

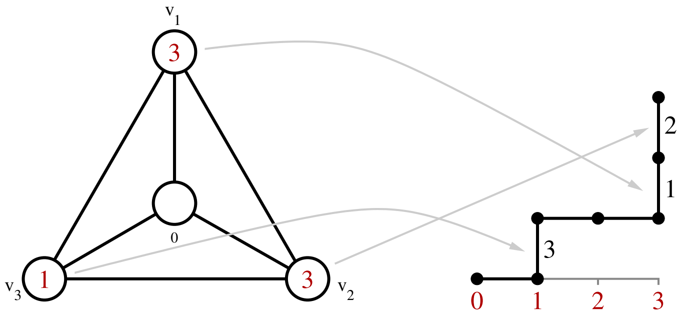
Chip-firing


Parking Functions
Rational Parking Functions
Chip Firing

CF: Chip configurations



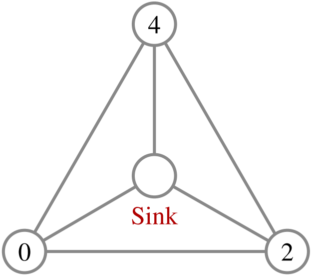
CF: Configurations as labeled paths



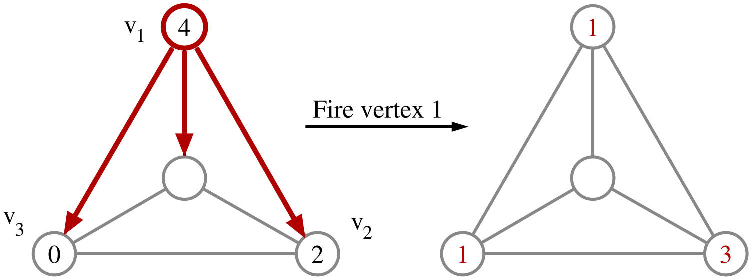
Configurations $\leftrightarrow$ Labeled lattice paths

Q: What's the connection with parking functions?

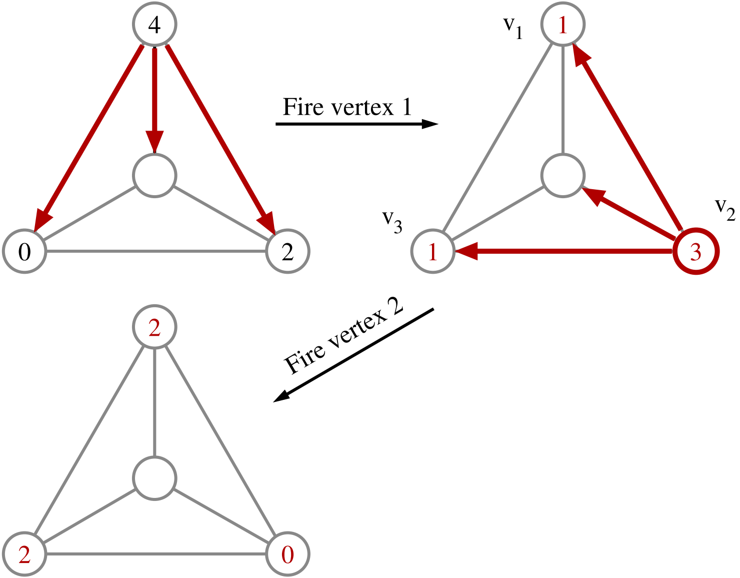
CF: Firing example



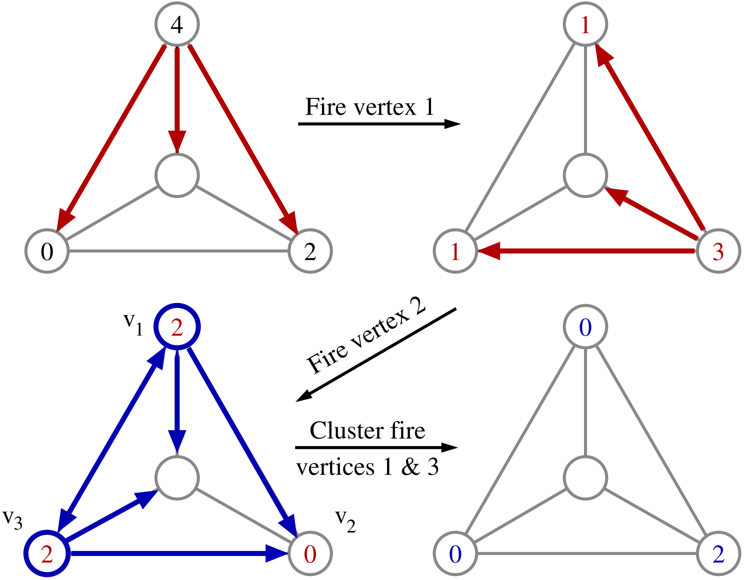
CF: Firing example



CF: Firing example

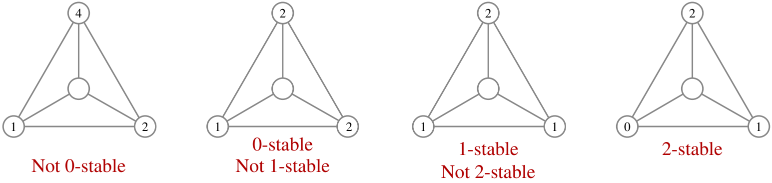


CF: Firing example



CF: Stability

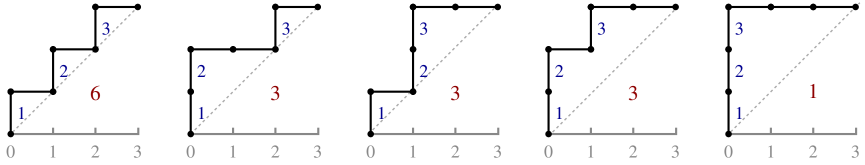
A nonnegative configuration is k-stable if there is no cluster of size at most $k+1$ that can legally fire.



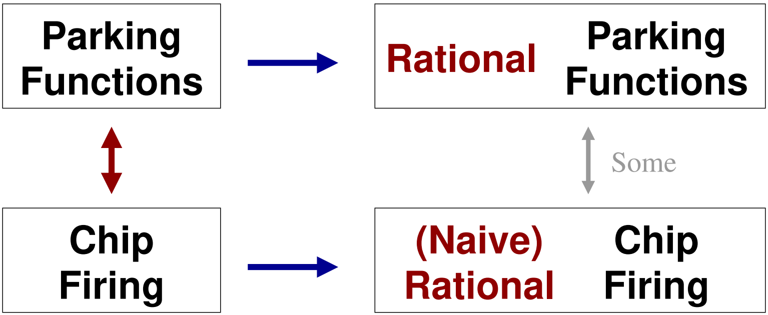
Historical focus has been on $k=0$ ("stable") and $k=n-1$ ("superstable") cases.

Fact: For $G = K_{n+1}$, the $(n-1)$-stable configurations are the parking functions.

- *G*-parking functions of Postnikov & Shapiro '04,
- *q*-reduced divisors of Baker & Norine '07.



Rational chip-firing?



Potential "rational" firing rule:

When fired, a chip sends:

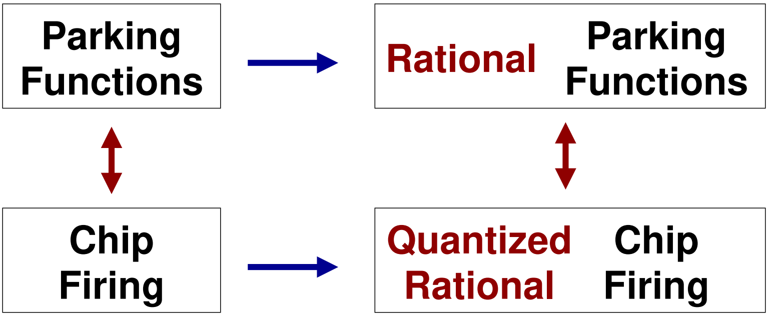
- One chip to the sink and
- a/b chips to each non-sink neighbor.

Note: Number of chips doesn't remain in $\mathbb{Z}$.

See related work:

(e.g., Backman-Charbonneau-Loehr-Mullins-O'Connor-W '24; Stanley-Wang '18; Postnikov-Shapiro '03).

Quantized rational chip-firing

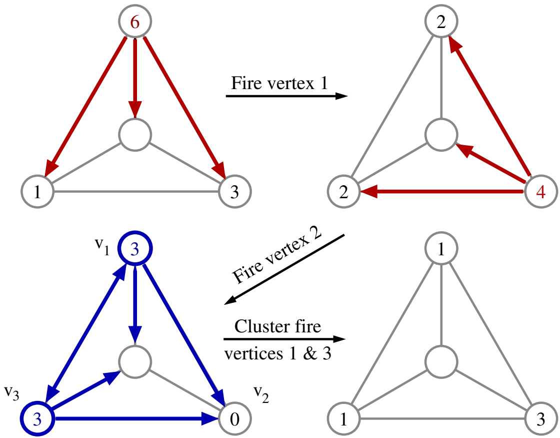


When a subset S of size s fires, each vertex

- in S loses $\left\lfloor (n-s)\frac{a}{b}+1 \right\rfloor$ chips,
- not in S gains $\left\lfloor s\frac{a}{b} \right\rfloor$ chips.

Note: This works on any* graph.

QRCF: Firing example for $a/b = 5/3$



Firing 1: $-\lfloor 2 \cdot 5/3 + 1 \rfloor = -4; \quad +\lfloor 1 \cdot 5/3 \rfloor = +1.$

Firing 2: $-\lfloor 1 \cdot 5/3 + 1 \rfloor = -2; \quad +\lfloor 2 \cdot 5/3 \rfloor = +3.$

QRCF: Is it any good...

- as a model of chip-firing?
- as an analogue of
  rational parking functions?

Theorem (Backman-Loehr-W '25):

- For $0 \leq k \leq n-1$, any nonnegative configuration has at least one k-stabilization;
- The 0-stabilization is unique for all G; and
- The $(n-1)$-stabilization is unique for $G = K_{n+1}$.

The rational correspondence

Theorem (BLW '25): For $\gcd(a,b)=1$, $G=K_{b+1}$,

$$\boxed{\textbf{Rational Parking Functions}}$$

$$\Updownarrow$$

$$\boxed{\textbf{(b-1)-stable Chip Configurations}}$$

Define $\mathrm{ch} \approx \mathrm{ch}'$ if they are related by a sequence of firings and/or borrowings.

Theorem (BLW '25): For $\gcd(a, b) = 1$, $G = K_{b+1}$, the critical group $\mathbb{Z}^b/\approx$ is isomorphic to $\mathbb{Z}_a^{b-1}$.

k-skeletal configurations

A configuration ch is k-skeletal if:

(C0) $\mathrm{ch} \geq 0$.

(C1) For all $S \subseteq [n]$, if $0 < |S| \leq k+1$, then S cannot legally fire in configuration ch.

(C2) For all nonempty $T \subseteq [n]$, if T can legally "borrow" in configuration ch, then there exists $S \subseteq [n]$ such that $0 < |S| \leq k+1$ and S can legally fire in the post-borrow configuration $\beta_T(\mathrm{ch})$.

For $k = 0$, we get recurrent (critical) configurations.

For $k = n-1$ we get superstable configurations.

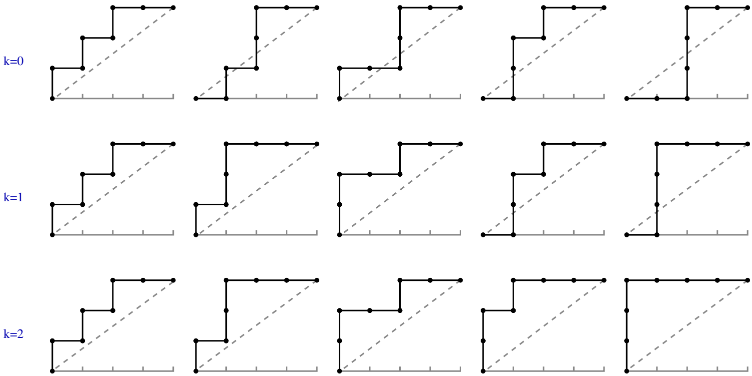
Define a path Q to be k-skeletal if

(R1) The last $k+1$ north steps of Q start on or above $bx = ay$.

(R2) For each lattice point v on Q with v strictly above $bx = ay$, Condition (R1) is false for Q_v.

Theorem (BLW '25): A path is k-skeletal iff the corresponding configuration is.

k-skeletal paths ($a = 4$, $b = 3$)



Theorem (BLW '25): For $\gcd(a, b) = 1$, $G = K_{b+1}$, the number of k-skeletal paths equal to a^{b-1} for all $0 \le k \le b - 1$.

Duality

- $\kappa = a - \lfloor a/b \rfloor - 1$,
- $C_0 = \{$0-skeletal configurations$\}$,
- $C_{b-1} = \{(b-1)$-skeletal configurations$\}$, and
- $\theta(\mathrm{ch})(i) = \kappa - \mathrm{ch}(i)$ for all i.

Theorem (BLW '25): For $\gcd(a, b) = 1$, $G = K_{b+1}$, $\theta : C_{b-1} \to C_0$ is a bijection.

This duality in the classical case is related to Riemann-Roch duality for graphs and Alexander duality for monomial ideals.

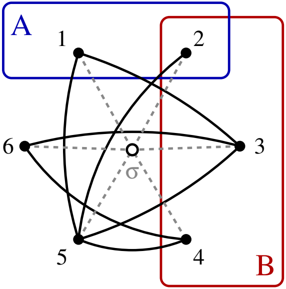
https://arxiv.org/abs/2603.15451

Danke Schön

QRCF: Non-uniqueness

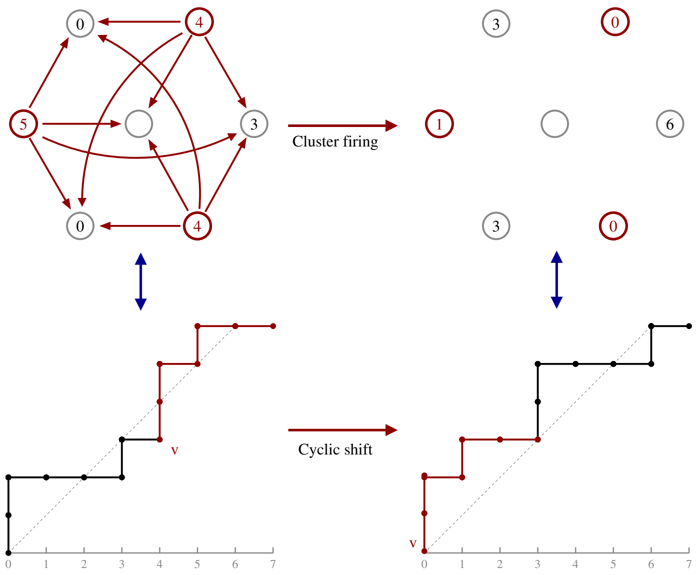
Example: The k-stabilization is not unique in general.

Let $a/b = 2/5$.



Vertex	1	2	3	4	5	6
ch	1	1	2	1	0	0
$\phi_A(\mathrm{ch})$	0	0	2	1	0	0
$\phi_{B\setminus A}(\phi_A(\mathrm{ch}))$	0	0	0	0	0	0
$\phi_B(\mathrm{ch})$	1	0	0	0	1	0
$\phi_{A\setminus B}(\phi_B(\mathrm{ch}))$	0	0	0	0	1	0

Motivation: sometimes classic firing corresponds to a cyclic shift



Motivation: almost works in classic case