

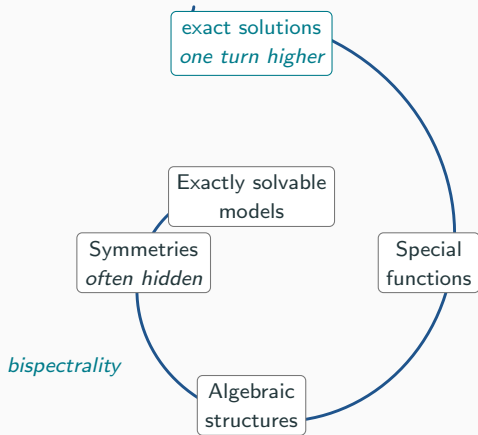
Algebras and Bispectrality

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The virtuous spiral



- Exact solvability rests, empirically, on *symmetries* — which are often hidden.
- Symmetries are described by **algebraic structures**.
- Their representations produce the *special functions* in which solutions are expressed.

Bispectrality: one function, two eigenvalue problems

$\psi_n(x)$ depends on two variables x and n — continuous or discrete — and solves an eigenvalue problem in *each*:

acting on x (eigenvalue $\Lambda(n)$)

$$A \psi_n(x) = \Lambda(n) \psi_n(x)$$

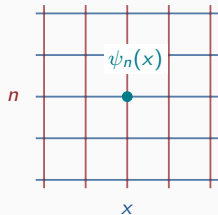
acting on n (eigenvalue $\theta(x)$)

$$A^* \psi_n(x) = \theta(x) \psi_n(x)$$

Two spectral equations for one function.

In the x -representation A^* is multiplication by $\theta(x)$;

A and A^* do *not* commute and generate the algebras we want to study and generalize.



one joint eigenfunction
on the (n, x) lattice

The goal of this talk

- i. give an **overview** of the algebraic theory of bispectrality;
- ii. describe some of the **bispectral algebras**;
- iii. cover their **relations to special functions**;
- iv. illustrate some **uses and applications** of these algebras (some other time).

Classical orthogonal polynomials are bispectral

Let $\{P_n(x)\}$ be orthogonal and *monic*, $\deg P_n = n$.

Orthogonality $\Rightarrow$ a first eigenvalue problem

By Favard, orthogonality is equivalent to a three-term recurrence

$$x P_n(x) = P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x).$$

“Classical” $\Rightarrow$ a second eigenvalue problem

The polynomials are *classical* when they also satisfy a difference/differential equation

$$D P_n(x) = \lambda_n P_n(x), \quad D \text{ acting on } x.$$

Two eigenvalue problems: the classical polynomials are **bispectral**.

Notation: hypergeometric and q -hypergeometric series

With the **Pochhammer** and **q -Pochhammer** symbols

$$(a)_k = \prod_{j=0}^{k-1} (a+j), \quad (a; q)_k = \prod_{j=0}^{k-1} (1 - aq^j),$$

the (basic) hypergeometric series read

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z \right) = \sum_{k \geq 0} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!},$$
$${}_{r+1}\phi_r \left(\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z \right) = \sum_{k \geq 0} \frac{(a_1; q)_k \cdots (a_{r+1}; q)_k}{(q; q)_k (b_1; q)_k \cdots (b_r; q)_k} z^k.$$

A **numerator** parameter equal to $-N$ (or q^{-N}) **truncates** the series to a polynomial of degree N — as $(-N)_k = 0$ for $k > N$; this yields the finite families used throughout.

The Racah polynomials

The example at the *top* of the $q = 1$ discrete Askey scheme: four real parameters $\alpha, \beta, \gamma, \delta$, degree n in the quadratic grid $\lambda(x) = x(x + \gamma + \delta + 1)$.

Explicit definition

$$R_n(\lambda(x)) = {}_4F_3 \left(\begin{matrix} -n, & n + \alpha + \beta + 1, & -x, & x + \gamma + \delta + 1 \\ & \alpha + 1, & \beta + \delta + 1, & \gamma + 1 \end{matrix} ; 1 \right), \quad n = 0, 1, 2, \dots$$

The series **terminates** thanks to the negative integers $-n$ and $-x$ in the numerator — so R_n is a polynomial of degree n in $\lambda(x)$.

Under a truncation — e.g. $\alpha + 1 = -N$ — they obey a *finite* orthogonality

$$\sum_{x=0}^N w_x R_n R_m = h_n \delta_{nm}.$$

$R_n(\lambda(x))$, $\lambda(x) = x(x+\gamma+\delta+1)$, is a joint eigenfunction of *two* operators — a recurrence operator $\mathcal{M}$ (in the degree n) and a difference operator $\mathcal{L}$ (in x , with $T^\pm f(x) = f(x\pm 1)$):

Recurrence operator $\mathcal{M}$ — action in n , eigenvalue $\lambda(x)$

$$\mathcal{M} R_n = A_n R_{n+1} - (A_n + C_n) R_n + C_n R_{n-1} = \lambda(x) R_n$$

$$A_n = \frac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(n+\gamma+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad C_n = \frac{n(n+\alpha+\beta-\gamma)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}$$

Difference operator $\mathcal{L}$ — action in x , eigenvalue $n(n+\alpha+\beta+1)$

$$\mathcal{L} R_n = [B(x) T^+ + D(x) T^- - (B+D) I] R_n = n(n+\alpha+\beta+1) R_n$$

$$B(x) = \frac{(x+\alpha+1)(x+\beta+\delta+1)(x+\gamma+1)(x+\gamma+\delta+1)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)}, \quad D(x) = B(x)|_{x \rightarrow -x-\gamma-\delta-1}$$

Two representations

In a representation, denote the pair by K_1, K_2 .

	on functions of x	on functions of n
K_1	mult. by $\lambda(x) = x(x+\gamma+\delta+1)$	recurrence op. $\mathcal{M}$ (three-term in n)
K_2	difference op. $\mathcal{L}$ (in x)	mult. by $n(n+\alpha+\beta+1)$

The Racah *duality* $x \leftrightarrow n$, $\alpha \leftrightarrow \gamma$, $\beta \leftrightarrow \delta$ swaps the two columns — i.e. interchanges $K_1 \leftrightarrow K_2$.

The two operators generate a quadratic algebra: Racah

Take the two bispectral operators as K_1, K_2 and set $K_3 = [K_1, K_2]$. Computing $[K_2, K_3]$ and $[K_3, K_1]$ shows that the bispectral operators realize the **Racah algebra**:

$$[K_1, K_2] = K_3,$$

$$[K_2, K_3] = K_2^2 + \{K_1, K_2\} + d K_2 + e_1,$$

$$[K_3, K_1] = K_1^2 + \{K_1, K_2\} + d K_1 + e_2,$$

The structure constants are given in terms of $\alpha, \beta, \gamma, \delta$:

$$e_1 = \frac{1}{4} \left(\frac{\alpha - \beta}{2} \right) \left(\frac{\alpha + \beta}{2} \right) \left(\frac{\alpha + \beta}{2} - \gamma \right) \left(\frac{\alpha - \beta}{2} - \delta \right), \quad e_2 = \frac{1}{4} \left(\frac{\gamma - \delta}{2} \right) \left(\frac{\gamma + \delta}{2} \right) \left(\frac{\gamma + \delta}{2} - \alpha \right) \left(\frac{\gamma - \delta}{2} - \beta \right),$$

$$d = \frac{1}{4} \left[\left(\frac{\gamma - \delta}{2} \right)^2 + \left(\frac{\gamma + \delta}{2} \right)^2 + \left(\frac{\gamma + \delta}{2} - \alpha \right)^2 + \left(\frac{\gamma - \delta}{2} - \beta \right)^2 - 2 \right].$$

Hahn: the linear-grid contraction

On the *linear* grid $\lambda(x) = x$ sit the Hahn polynomials $Q_n(x; \alpha, \beta, N)$ — one step down the Askey scheme — on $x \in \{0, \dots, N\}$. They too are bispectral:

$$\begin{aligned}\mathcal{M} Q_n &= -A_n Q_{n+1} + (A_n + C_n) Q_n - C_n Q_{n-1} = -x Q_n, \\ A_n &= \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad C_n = \frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \\ \mathcal{L} Q_n &= B(x) Q_n(x+1) - (B+D) Q_n(x) + D(x) Q_n(x-1) = n(n+\alpha+\beta+1) Q_n, \\ B(x) &= (x+\alpha+1)(x-N), \quad D(x) = x(x-\beta-N-1).\end{aligned}$$

The same computation now closes into the **Hahn algebra**. Unlike Racah, the relations are *not* symmetric under $K_1 \leftrightarrow K_2$, since the polynomials are not self-dual (under $x \leftrightarrow n$ one obtains the **dual Hahn** polynomials):

$$[K_1, K_2] = K_3, \quad [K_2, K_3] = \{K_1, K_2\} + d_1 K_1 + d_2, \quad [K_3, K_1] = K_1^2 + d_3 K_2 + d_4.$$

Leonard pair: encapsulating bispectrality cases

Definition (Terwilliger)

An ordered pair (A, A^*) of operators on a finite-dimensional V is a **Leonard pair** if

- (i) in some basis A is diagonal and A^* is irreducible tridiagonal;
- (ii) in some basis A^* is diagonal and A is irreducible tridiagonal,

where *irreducible tridiagonal* means all sub- and super-diagonal entries are nonzero.

basis where A is diagonal

$$A: \begin{bmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{bmatrix} \quad A^*: \begin{bmatrix} \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \\ & \bullet & \bullet & \bullet \\ & & \bullet & \bullet \end{bmatrix}$$

basis where A^* is diagonal

$$A: \begin{bmatrix} \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ & \bullet & \bullet & \bullet \\ & & \bullet & \bullet \end{bmatrix} \quad A^*: \begin{bmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{bmatrix}$$

The overlap of the two eigenbases is bispectral

Self-adjoint pair: eigenbases $|\theta_i\rangle, |\theta_j^*\rangle$ orthonormal, overlap $\phi(i, j) = \langle \theta_i | \theta_j^* \rangle$. Evaluate one matrix element *two ways*.

$\langle \theta_i | A^* | \theta_j^* \rangle$ **two ways** $\Rightarrow$ **three-term recurrence in i**

A^* on the right gives $\theta_j^* \phi$; on the left (A^* tridiagonal in $|\theta_i\rangle$):

$$\theta_j^* \phi(i, j) = \beta_i \phi(i+1, j) + \alpha_i \phi(i, j) + \gamma_i \phi(i-1, j)$$

$\langle \theta_i | A | \theta_j^* \rangle$ **two ways** $\Rightarrow$ **three-term recurrence in j**

A on the left gives $\theta_i \phi$; on the right (A tridiagonal in $|\theta_j^*\rangle$):

$$\theta_i \phi(i, j) = \tilde{\beta}_j \phi(i, j+1) + \tilde{\alpha}_j \phi(i, j) + \tilde{\gamma}_j \phi(i, j-1)$$

$\Rightarrow$ the overlap $\phi(i, j)$ is **bispectral** in the two discrete variables i and j .

Duality of orthogonal polynomials

If a finite family of orthogonal polynomials has the property that its *dual* — obtained by exchanging the roles of degree and argument — is *again* a family of orthogonal polynomials, then it is a special or limiting case of the $(q-)$ Racah / Askey–Wilson polynomials.

- Leonard (1982) — the theorem, for polynomials orthogonal on a finite set;
- Bannai–Ito (1984) — the association-scheme context;
- Terwilliger (2001) — the linear-algebraic form: *Leonard pairs*.

It applies here: the definition is symmetric under $A \leftrightarrow A^*$, and the overlaps obey a three-term recurrence in *each* spectrum — by Favard, an orthogonal family whose *dual* is orthogonal too, exactly Leonard's hypothesis.

Tridiagonality constrains both spectra

Data of the pair: the eigenvalues $\{\theta_i\}_{i=0}^d$ of A and $\{\theta_i^*\}_{i=0}^d$ of A^* . Tridiagonality in *each other's* eigenbasis strongly constrains these numbers.

The constraint (Leonard 1982; Terwilliger 2001)

A single scalar $\beta = q + q^{-1}$ is *shared* by both sequences, each combination below being *independent of i* :

$$\theta_{i-1} - \beta \theta_i + \theta_{i+1}, \quad \theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^* \quad (1 \leq i \leq d-1).$$

So each spectrum takes the *q -quadratic grid* form

$$\theta_i = a + b q^i + c q^{-i}, \quad \theta_i^* = a^* + b^* q^i + c^* q^{-i} \quad (\rightarrow a + bi + ci^2 \text{ as } q \rightarrow 1)$$

— precisely the eigenvalue grid of the *q -Racah polynomials*.

Algebraic implication

What do the two properties of a Leonard pair impose on A and A^* ?

Work in the eigenbasis of A (diagonal, entries θ_i), with the q -commutator

$$[X, Y]_q := q^{1/2}XY - q^{-1/2}YX:$$

- a bracket with A acts *entrywise*: $([A, X]_q)_{ij} = (q^{1/2}\theta_i - q^{-1/2}\theta_j)X_{ij}$;
- so $[A, [A, A^*]_q]_{q^{-1}} = A^2A^* - \beta AA^*A + A^*A^2$ multiplies A^*_{ij} by $\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2$, with $\beta = q + q^{-1}$;
- tridiagonality leaves $j = i, i \pm 1$, and on the **q -quadratic grid** $\beta = q + q^{-1}$ cancels the $q^{\pm 2i}$ terms — what remains is affine in $\theta_i + \theta_j$, hence back on A, A^* :

$$A^2A^* - \beta AA^*A + A^*A^2 = (\text{quadratic in } A, A^*) \quad (\text{and the } A \leftrightarrow A^* \text{ partner}).$$

These are the **Askey–Wilson relations** — the algebra of the q -Racah polynomials.

Representations of the Askey–Wilson algebra

Take a finite-dimensional irreducible representation of the Askey–Wilson algebra, and the eigenbasis $|\theta_i\rangle$ of A .

- the relations make A^* **tridiagonal** in that basis — the **representation matrices**, with entries $A_{i,i}^*$ and $A_{i,i\pm 1}^*$ fixed by the structure constants;
- the relations are symmetric under $A \leftrightarrow A^*$, so dually A is tridiagonal in the eigenbasis $|\theta_j^*\rangle$;
- the overlaps $\phi(i,j) = \phi(0,j) P_i(\theta_j^*)$ then obey the three-term recurrence

$$\theta_j^* P_i(\theta_j^*) = A_{i,i+1}^* P_{i+1}(\theta_j^*) + A_{i,i}^* P_i(\theta_j^*) + A_{i,i-1}^* P_{i-1}(\theta_j^*),$$

whose coefficients — the very entries of A^* — are **exactly the recurrence coefficients of the q -Racah polynomials**. Hence the P_i are the q -Racah polynomials.

The q -Racah polynomials

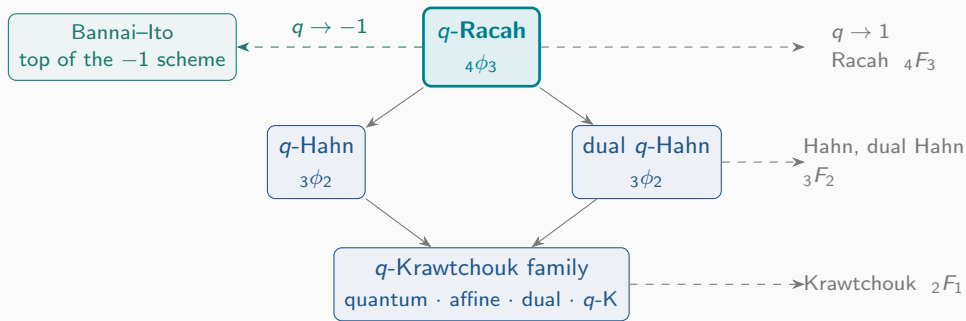
The q -analogue of the Racah ${}_4F_3$ met above — same four parameters, degree n in the q -quadratic grid $\mu(x) = q^{-x} + \gamma\delta q^{x+1}$:

The same pattern, q -deformed

$$R_n(\mu(x)) = {}_4\phi_3 \left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \gamma\delta q^{x+1} \\ \alpha q, \beta\delta q, \gamma q \end{matrix} \middle| q, q \right)$$

They sit at the **top** of the finite part of the q -Askey scheme.

Classification: the finite Askey scheme



A *generic* Leonard pair gives q -Racah. To classify *all* of them: they are in **correspondence with the families obtained as special cases or limits of q -Racah** — and only the **finite** ones occur.

The table *is* that classification. **Bannai-Ito** ($q \rightarrow -1$): we shall come back to it.

Structure and algebraic properties

Abelian

A, A^* diagonal in a *common* basis $\iff [A, A^*] = 0$.

Quadratic

The next level of complexity: A, A^* one diagonal, the other tridiagonal and vice-versa — the Leonard pair conditions — implies a **quadratic algebra**,

$$[A, [A, A^*]_q]_{q^{-1}} = (\text{quadratic in } A, A^*), \quad (\text{and the } A \leftrightarrow A^* \text{ partner}).$$

This is the **Askey–Wilson algebra**; at $q \rightarrow 1$ ($[X, Y]_q \rightarrow [X, Y]$), the **Racah algebra** found above.

Association schemes and graphs

A (symmetric) **association scheme** on a finite set X is a family of **adjacency matrices** $A_0, A_1, \dots, A_D \in \text{Mat}_X(\mathbb{C})$ — symmetric $\{0, 1\}$ -matrices, one per relation — obeying

Axioms

$$A_0 = \mathbb{I}, \quad \sum_{i=0}^D A_i = J \text{ (all-ones matrix)}, \quad A_i A_j = \sum_{k=0}^D p_{ij}^k A_k = A_j A_i.$$

- closure (integer **intersection numbers** p_{ij}^k) $\Rightarrow$ the span is a *commutative algebra*, the **Bose–Mesner algebra** $\mathcal{M}$;
- classifying association schemes is a central, largely open problem; the P/Q -polynomial ones — our concern — are the most structured corner.

The primitive idempotents

The A_i are real symmetric and commute, so they share one set of orthogonal spectral projectors — the **primitive idempotents** $E_0, \dots, E_D$ onto the common eigenspaces. Every A_i has the **spectral decomposition**

$$A_i = \sum_{j=0}^D p_i(j) E_j, \quad E_j E_l = \delta_{jl} E_j, \quad \sum_{j=0}^D E_j = \mathbb{I}.$$

So $\{A_i\}$ and $\{E_j\}$ are two bases of the algebra $\mathcal{M}$, related by the **eigenvalue matrix** $p_i(j)$.

P - and Q -polynomial

Fix a base vertex x_0 . The **dual adjacency** matrices A_i^* and the **dual idempotents** E_i^* are defined by

$$[A_i^*]_{xx} = |X| [E_i]_{x_0x}, \quad [E_i^*]_{xx} = [A_i]_{x_0x} = \delta_{d(x_0,x), i}.$$

P -polynomial

every A_i is a degree- i polynomial in $A \equiv A_1$
 $\Rightarrow A$ alone generates $\mathcal{M}$ — a
distance-regular graph.

Q -polynomial

every A_i^* is a degree- i polynomial in $A^* \equiv A_1^*$
 $\Rightarrow A^*$ alone generates the dual $\mathcal{M}^*$.

A is constant on each *eigenspace* $\text{Im } E_i$; A^* on each *shell* $\text{Im } E_i^*$ (distance from x_0).

The Terwilliger algebra of the scheme

The **Terwilliger algebra** is generated by *both* the adjacency and the dual adjacency matrices:

$$T = \langle A_0, \dots, A_D, A_0^*, \dots, A_D^* \rangle \stackrel{P, Q\text{-poly}}{=} \langle A, A^* \rangle \quad (\text{noncommutative, finite-dim.}).$$

It is noncommutative because A (adjacency) and A^* (distance from x_0) do not commute.

Inside it:

- A^* **diagonal** in the vertex basis; an edge moves a vertex only to a *neighbouring* shell, so A is **tridiagonal** ($E_j^* A E_i^* = 0$ for $|i - j| > 1$: the P -polynomial property);
- dually (Q -polynomial): A **diagonal** in the spectral basis, A^* **tridiagonal** there.

Diagonal in one basis, tridiagonal in the other — the **Leonard-pair** shape. But the shells are in general higher-dimensional, so A is only *block*-tridiagonal; one condition — thinness (next) — collapses each shell to a single state.

Thin modules: the exact link to Leonard pairs

Decompose the vertex space into irreducible T -modules. On a module W , A^* acts as the scalar θ_i^* on each shell $E_i^* W$ — diagonal, but with *multiplicities* when $\dim E_i^* W > 1$, while A is *block*-tridiagonal.

Thin (Terwilliger)

W is **thin** if $\dim E_i^* W \leq 1$ for every shell i — at most one state per shell.

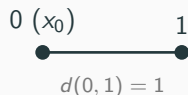
Then each θ_i^* occurs once, so A^* is **multiplicity-free** and A **irreducible tridiagonal**:

thin irreducible T -module $\iff (A, A^*)$ is a **Leonard pair**.

- on a P - and Q -polynomial scheme the $(D+1)$ -dimensional irrep — *one state per shell* — is a Leonard pair, its overlaps the scheme's $(q-)$ Racah-type polynomials.

The Terwilliger algebra of the (binary) Hamming scheme

Complete graph K_2 :



Adjacency (the edge), and dual adjacency:

$$A \equiv A_1 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^* = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A^* from the definition $[A^*]_{xx} = |X| [E_1]_{x_0 x}$ with $E_1 = \frac{1}{2}(\mathbb{I} - \sigma^x)$, $x_0 = 0$, $|X| = 2$.

σ^x, σ^z are the **Pauli matrices**; A, A^* generate the $\text{spin-}\frac{1}{2}$ representation of **$\mathfrak{su}(2)$** .

The hypercube $H(d, 2) = K_2^{\square d}$

- Cartesian product of d copies of K_2 ; vertices = length- d binary strings; Hamming distance $d(x, y) = \#\{i : x_i \neq y_i\}$;
- **Tensor notation:** $\sigma_i^\alpha = \mathbb{I} \otimes \cdots \otimes \sigma_i^\alpha \otimes \cdots \otimes \mathbb{I}$;
- $A \equiv A_1 = \sum_{i=1}^d \sigma_i^x$;
 $A^* = \sum_{i=1}^d \sigma_i^z$ (diagonal, reads distance from 0^d);
- A, A^* generate the **direct-product representation** $(\frac{1}{2})^{\otimes d}$ of $\mathfrak{su}(2)$;
the Terwilliger algebra of the Hamming scheme is $\mathfrak{su}(2)$.

A Leonard pair on each irreducible block

- $A = 2J_x$, $A^* = 2J_z$ act on all of $(\mathbb{C}^2)^{\otimes d}$; the Clebsch–Gordan similarity transformation performs its decomposition into irreducible spin- j representations;
- each spin- j irrep has dimension $2j + 1$, basis $\{|j, m\rangle\}$ — joint eigenvectors of $\mathbf{J}^2$ (eigenvalue $j(j+1)$) and J_z (eigenvalue $m = -j, -j+1, \dots, j$);
- *restricted to an irrep*, $A^*|j, m\rangle = 2m|j, m\rangle$ (diagonal) while A is tridiagonal,

$$A|j, m\rangle = \sqrt{(j-m)(j+m+1)}|j, m+1\rangle + \sqrt{(j+m)(j-m+1)}|j, m-1\rangle;$$

- so the restriction of (A, A^*) is a Leonard pair, with Krawtchouk overlaps.

The Johnson scheme $J(n, k)$

- vertices = the k -subsets of $\{1, \dots, n\}$, so $|X| = \binom{n}{k}$;
Johnson distance $d(x, y) = k - |x \cap y|$;
adjacency $A \equiv A_1$ (subsets sharing $k - 1$ elements);
- its combinatorics makes it **metric**: the Bose–Mesner product closes on neighbours,
 $A_1 A_i = p_{1i}^{i-1} A_{i-1} + p_{1i}^i A_i + p_{1i}^{i+1} A_{i+1}$ — a **three-term recurrence**;
- hence every $A_i = v_i(A)$ is a degree- i polynomial in A : the scheme is **P -polynomial**, and
dually **Q -polynomial** — like Hamming, so the general Leonard-pair theory applies.

$J(n, k)$ sits in the hypercube $H(n, 2)$

- a k -subset = a length- n binary string with exactly k ones — a **weight- k string**; e.g. $\{1, 3\} \subset \{1, 2, 3, 4\} \leftrightarrow 1010$;
- the **weight- k shell** E_k^* of $H(n, 2)$ is the set of all weight- k strings — those at Hamming distance $\partial = k$ from 0^n (recall $\partial(u, v) = \#\{i : u_i \neq v_i\}$);
- so $J(n, k)$ is this shell; a one-element swap flips *two* bits, so the Johnson distance is half the Hamming one: $d = \frac{1}{2}\partial$ (e.g. $\{1, 3\} \rightarrow \{1, 4\}$: $1010 \rightarrow 1001$, $\partial = 2$, $d = 1$).

Split by x_0 : two hypercubes, two $\mathfrak{su}(2)$

Running example: $n = 5$, $k = 2$, reference $x_0 = \{1, 2\}$.

- fix x_0 ; split the n sites into the k inside x_0 and the $n - k$ outside (here $\{1, 2\}$ and $\{3, 4, 5\}$);
- the bits of each group form a sub-hypercube, $H(k, 2)$ and $H(n - k, 2)$ — here a square $H(2, 2)$ and a cube $H(3, 2)$ — each with its own $\mathfrak{su}(2)$: $\mathbf{J}_2$ on x_0 , $\mathbf{J}_1$ outside;
- a weight- k string splits its k ones between the blocks — e.g. $\{1, 3\}$ (at distance 1 from x_0) = $\underbrace{10}_{\text{square}} \underbrace{100}_{\text{cube}}$, one 1 each — so $T(J(n, k))$ lives in two commuting $\mathfrak{su}(2)$, $\mathbf{J}_1 \oplus \mathbf{J}_2$.

The two generators A_J, A_J^* — and why

- **dual adjacency**: A^* reads the distance to the base vertex — to 0^n that is $\propto J_1^z + J_2^z$; to x_0 (all-1 on its k sites) the inside reference flips sign, so $A_J^* \propto J_1^z - J_2^z$ — diagonal in the **uncoupled** basis $|j_1 m_1; j_2 m_2\rangle$;
- **adjacency** A_J : a one-element swap transfers a spin between the blocks, $J_1^+ J_2^- + J_1^- J_2^+$; by

$$\mathbf{J}_1 \cdot \mathbf{J}_2 = J_1^z J_2^z + \frac{1}{2} (J_1^+ J_2^- + J_1^- J_2^+),$$

this is the off-diagonal part of the **coupling** $\mathbf{J}_1 \cdot \mathbf{J}_2$; with $\mathbf{J}_1^2, \mathbf{J}_2^2$ central, $A_J \propto (\mathbf{J}_1 + \mathbf{J}_2)^2$ (total Casimir), diagonal in the **coupled** basis $|j, m\rangle$.

The Terwilliger algebra is quadratic: the Hahn algebra

Recall $A_J^* \propto J_1^z - J_2^z$ and $A_J \propto (\mathbf{J}_1 + \mathbf{J}_2)^2$ (total Casimir); set $K_1 \propto A_J^*$, $K_2 = A_J$, $K_3 = [K_1, K_2]$.

Using the $\mathfrak{su}(2)$ algebra, one sees that these close into a **quadratic** algebra — the **Hahn algebra**:

$$[K_2, K_3] = \{K_1, K_2\} + d_1 K_1 + d_2, \quad [K_3, K_1] = K_1^2 + d_3 K_2 + d_4.$$

The Terwilliger algebra of $J(n, k)$ is the **quadratic Hahn algebra**.

The functions: dual Hahn polynomials

- the representation theory of the Hahn algebra produces these polynomials — and the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ operator realization makes it manifest;
- coupled $\leftrightarrow$ uncoupled is done by the Clebsch–Gordan matrix, whose elements are the dual Hahn polynomials, $\langle j, m | j_1 m_1; j_2 m_2 \rangle \propto R_i(\lambda(x); \gamma, \delta, N)$;
- next we couple three copies of $\mathfrak{su}(2)$ (the triple product) to reach Racah.

The Racah algebra from three $\mathfrak{su}(2)$

Recall $\mathfrak{su}(2) = \langle J_z, J_{\pm} \rangle$, with **Casimir** C :

$$[J_z, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_z, \quad C = J_z^2 + J_+ J_- - J_z.$$

Take three irreducible representations, $C^{(i)} = \lambda_i$ ($i = 1, 2, 3$), and form $J^{(123)} = J^{(1)} + J^{(2)} + J^{(3)}$ ($J^{(1)} = J \otimes I \otimes I$, etc.), carrying the **total Casimir** $C^{(123)} = \lambda_{123}$. The **intermediate Casimirs**

$$C^{(ij)} = [J_z^{(i)} + J_z^{(j)}]^2 + (J_+^{(i)} + J_+^{(j)})(J_-^{(i)} + J_-^{(j)}) - (J_z^{(i)} + J_z^{(j)}),$$

$(ij) = (12), (23), (31)$, are tied to the initial and total ones by

$$C^{(123)} = C^{(12)} + C^{(23)} + C^{(31)} - C^{(1)} - C^{(2)} - C^{(3)}.$$

The intermediate Casimirs close into the Racah algebra

Set $K_1 = -\frac{1}{2}C^{(12)}$, $K_2 = -\frac{1}{2}C^{(23)}$, $K_3 = [K_1, K_2]$; a direct computation returns the **Racah algebra** (Genest, V, Zhedanov):

$$[K_1, K_2] = K_3,$$

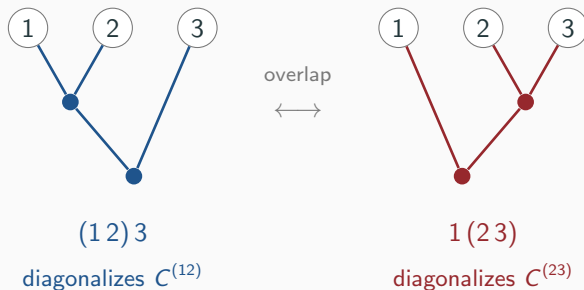
$$[K_2, K_3] = K_2^2 + \{K_1, K_2\} + d K_2 + e_1,$$

$$[K_3, K_1] = K_1^2 + \{K_1, K_2\} + d K_1 + e_2,$$

its three structure constants fixed by the initial and total Casimirs $\lambda_1, \lambda_2, \lambda_3, \lambda_{123}$:

$$d = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_{123}), \quad e_1 = \frac{1}{4}(\lambda_1 - \lambda_{123})(\lambda_2 - \lambda_3), \quad e_2 = \frac{1}{4}(\lambda_1 - \lambda_2)(\lambda_{123} - \lambda_3)$$

The recoupling coefficients are the Racah polynomials



The two eigenbases — of $C^{(12)}$ and $C^{(23)}$, i.e. of K_1 and K_2 — form a **Leonard pair**; their overlap is the $\mathfrak{su}(2)$ Racah coefficient (a $6j$ symbol):

$$\langle (12)3 | 1(23) \rangle \propto R_n(\lambda(x); \alpha, \beta, \gamma, \delta).$$

Bannai–Ito: the $q = -1$ case of the classification

In their classification of P - and Q -polynomial association schemes based on Leonard's theorem, Bannai and Ito found an additional case, corresponding to the $q = -1$ limit of the q -Racah polynomials — the Bannai–Ito polynomials.

The bispectrality of these polynomials was not understood for a long time; it was resolved about ten years ago.

Bispectrality and Dunkl operators (Tsujiimoto, V, Zhedanov)

Alongside their three-term recurrence, the BI polynomials are eigenfunctions of a first-order **Dunkl shift operator** L — built from a shift *and* the reflection R , $Rf(x) = f(-x)$:

$$L = F(x)(1 - R) + G(x)(T^+R - 1), \quad LP_n = \lambda_n P_n,$$

with $T^+f(x) = f(x + 1)$ (so $T^+Rf(x) = f(-x - 1)$) and

$$F(x) = \frac{(x - \rho_1)(x - \rho_2)}{x}, \quad G(x) = \frac{(x - r_1 + \frac{1}{2})(x - r_2 + \frac{1}{2})}{x + \frac{1}{2}} \quad (\rho_1, \rho_2, r_1, r_2 \in \mathbb{R}).$$

This is the most general first-order operator in shift and reflection preserving each polynomial degree.

The Bannai–Ito algebra

L and multiplication by x realize the Bannai–Ito algebra, with defining relations

$$\{K_1, K_2\} = K_3 + \omega_3,$$

$$\{K_2, K_3\} = K_1 + \omega_1,$$

$$\{K_3, K_1\} = K_2 + \omega_2,$$

$$(\{A, B\} = AB + BA, \text{ Casimir } Q = K_1^2 + K_2^2 + K_3^2).$$

A realization can also be found in terms of intermediate Casimirs — the three-fold product of $\mathfrak{su}(2)$ replaced by that of the superalgebra $\mathfrak{osp}(1|2)$.

Higher rank: coupling n copies of $\mathfrak{su}(2)$

In order to define higher-rank algebras of Askey–Wilson type, one approach is to extend to the n -fold tensor product the realization by intermediate Casimir elements.

Consider $\mathfrak{su}(2)^{\otimes n}$: for each subset $A \subseteq [n]$ the factors in A carry a coupled $\mathfrak{su}(2)$ representation, with the intermediate Casimir C_A as its Casimir. Two facts organize everything:

- any *three* disjoint blocks form a three-fold coupling, so their intermediate Casimirs satisfy the rank-one Racah relations;
- taken together, the C_A generate the higher-rank Racah algebra $\mathcal{R}(n)$: a web of rank-one Racah algebras, one for every triple of blocks.

The rank n Racah algebra

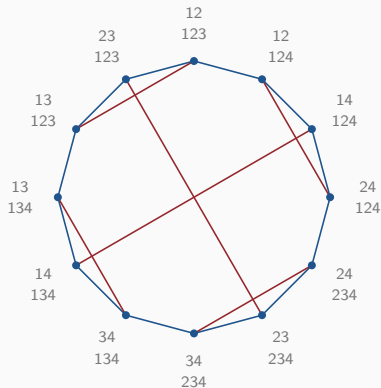
$$P_{ij} = C_{ij} - C_i - C_j, \quad F_{ijk} = \frac{1}{2}[P_{ij}, P_{jk}].$$

The defining relations on which $\mathcal{R}(n)$ closes (De Bie, Genest, van de Vijver & V):

$$\begin{aligned} [P_{ij}, P_{jk}] &= 2F_{ijk}, & [P_{kl}, F_{ijk}] &= P_{ik}P_{jl} - P_{il}P_{jk}, \\ [P_{jk}, F_{ijk}] &= P_{ik}P_{jk} - P_{jk}P_{ij} + 2P_{ik}C_j - 2P_{ij}C_k, \\ [F_{ijk}, F_{jkl}] &= F_{jkl}P_{ij} - F_{ikl}(P_{jk} + 2C_j) - F_{ijk}P_{jl}, & [F_{ijk}, F_{klm}] &= F_{ilm}P_{jk} - P_{ik}F_{jlm}. \end{aligned}$$

- blocks with no factor in common commute (last two relations: $n \geq 5$);
- a *central quotient* of $\mathcal{R}(n)$ is exactly the centralizer of the diagonal $\mathfrak{sl}_2$ in $U(\mathfrak{sl}_2)^{\otimes n}$ (Crampé, Gaboriaud, Poulain d'Andecy & V).

The recoupling graph: assembling the overlaps



Vertices are the **bases** $\langle C_{ij}, C_{ijk} \rangle$; every **edge** is a single-Casimir change — one *univariate* Racah. The graph is **connected**, so any two bases are joined by a path and the overlap is a product of univariate Racah polynomials,

$$\prod_a R_{k_a}(\lambda_a; \dots),$$

the **Tratnik multivariate Racah polynomials**; $\mathcal{R}(n)$ is their bispectral algebra.

Multivariate P -polynomial schemes

A **significant development** in the (still largely open) classification of association schemes (Bernard, Crampé, Poulain d'Andecy, V, Zaimi): the theory of **multivariate P - (and Q -) polynomial schemes**.

- **Classical:** $A_i = v_i(A_1) \Leftrightarrow$ a **distance-regular graph** $\Leftrightarrow$ a one-variable (P -polynomial) scheme.
- **Multivariate:** $A_{\vec{n}} = v_{\vec{n}}(A_{e_1}, \dots, A_{e_m})$ defines a **multivariate P -polynomial scheme**, whose graph side is an **m -distance-regular graph** — one *vector* distance, m edge types.
- **An example:** the **ordered Hamming scheme** (of depth 2), whose Terwilliger algebra is $\mathfrak{su}(3)$ and whose adjacency matrices are defined by the **bivariate Krawtchouk polynomials** of Tratnik type.

The algebraic Heun operator (Grünbaum, V, Zhedanov)

To any pair of bispectral operators (X, Y) — one tridiagonal in the eigenbasis of the other — one defines as follows the **algebraic Heun operator**:

$$W = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0 I.$$

Tridiagonal in both eigenbases

For a **Leonard pair**, Nomura–Terwilliger proved W is the *unique* such operator.

Why “Heun”

Consider the **bispectral operators of the Jacobi polynomials**

$$X = x, \quad Y = x(1-x)\partial_x^2 + (\alpha_1 x + \alpha_2)\partial_x.$$

The bilinear operator in X and Y is

$$W = x(x-1)(x-d)\partial_x^2 + [\gamma(x-1)(x-d) + \delta x(x-d) + \varepsilon x(x-1)]\partial_x + (\alpha\beta x - q).$$

This is the **standard Heun operator**, which leads to the general second-order Fuchsian differential equation with regular singularities at $0, 1, d, \infty$.

This explains the name — when generalized to other bispectral situations.

- Orthogonal polynomials led us to the Leonard pair.
- We now extend the picture to bispectral biorthogonal rational functions $w_n(x)$ (BRF).
- These obey generalized eigenvalue problems (GEVPs) — Zhedanov, 1999.

Generalized eigenvalue problems $\rightarrow$ rational functions

A **generalized eigenvalue problem** (GEVP) — a *pencil* of two tridiagonal matrices (X, Z) ,
 $X w = \tilde{\lambda}_x Z w$, i.e.

$$X_{n+1,n} w_{n+1} + X_{n,n} w_n + X_{n-1,n} w_{n-1} = \tilde{\lambda}_x (Z_{n+1,n} w_{n+1} + Z_{n,n} w_n + Z_{n-1,n} w_{n-1}).$$

($Z = \mathbb{I}$ gives an ordinary eigenvalue problem.) Normalise $w_0 = 1$ and climb it — isolating the top term forces a **division**:

$$w_{n+1}(x) = \frac{(\tilde{\lambda}_x Z_{n,n} - X_{n,n}) w_n(x) + (\tilde{\lambda}_x Z_{n-1,n} - X_{n-1,n}) w_{n-1}(x)}{X_{n+1,n} - \tilde{\lambda}_x Z_{n+1,n}}.$$

- the denominator is *linear* in $\tilde{\lambda}_x$ — each step adds a **pole**, so w_n is a **rational function** of $\tilde{\lambda}_x$ of degree n .

The Leonard trio (Crampé, Groenevelt, Labriet, Morey, V, Wagenaar)

A **Leonard trio** $(V, \tilde{V}, Z)$ on a space of dimension $N+1$ admits *two* bases, in which:

basis (i): V diagonal

$$\begin{array}{l} V : \\ Z : \\ \tilde{V}Z : \end{array} \left[\begin{array}{c} \text{blue dots on diagonal} \\ \text{red dots on sub-diagonal} \\ \text{red dots on super-diagonal} \end{array} \right]$$

basis (ii): $\tilde{V}$ diagonal

$$\begin{array}{l} \tilde{V} : \\ Z : \\ ZV : \end{array} \left[\begin{array}{c} \text{blue dots on diagonal} \\ \text{red dots on sub-diagonal} \\ \text{red dots on super-diagonal} \end{array} \right]$$

- V and $\tilde{V}$ are diagonal (multiplicity free) in basis (i) and (ii) *respectively* — and may be *full* in the other basis;
- Z is irreducible tridiagonal in *both* bases;
- the object of interest is the overlap $w_n(x) = \langle \tilde{v}_x^* | v_n \rangle$ ($Vv_n = \lambda_n v_n$, $\tilde{V}\tilde{v}_x = \tilde{\lambda}_x \tilde{v}_x$).

Some properties

- **Affine freedom:** $V \rightarrow v_1 V + v_0 \mathbb{I}$, $\tilde{V} \rightarrow \tilde{v}_1 \tilde{V} + \tilde{v}_0 \mathbb{I}$ and the scaling $Z \rightarrow z_1 Z$ all keep $(V, \tilde{V}, Z)$ a trio.
- **Z is rigid:** it cannot be *shifted*, $Z \not\rightarrow Z + z_0 \mathbb{I}$. Then $\tilde{V}Z \rightarrow \tilde{V}Z + z_0 \tilde{V}$, and $\tilde{V}$ may be *full* in the V -basis, so $\tilde{V}Z$ would lose its tridiagonal shape.
- **Order matters:** $(V, \tilde{V}, Z)$ a trio does *not* imply $(\tilde{V}, V, Z)$ is one.

Simple observations, and the four tridiagonal actions

- if $(V, \tilde{V})$ is a **Leonard pair**, then $(V, \tilde{V}, \mathbb{I})$ is a trio;
- if (V, Z) is a **Leonard pair**, then (V, V, Z) is a trio (take $\tilde{V} = V$): V diagonal and Z tridiagonal, so VZ, ZV are tridiagonal too.

$$Vv_n = \lambda_n v_n, \quad \tilde{V}\tilde{v}_x = \tilde{\lambda}_x \tilde{v}_x$$

in the V -eigenbasis

$$\tilde{V}Z v_n = \sum_{k=n-1}^{n+1} X_{k,n} v_k$$

$$Z v_n = \sum_{k=n-1}^{n+1} Z_{k,n} v_k$$

in the $\tilde{V}$ -eigenbasis

$$ZV \tilde{v}_x = \sum_{y=x-1}^{x+1} \tilde{X}_{y,x} \tilde{v}_y$$

$$Z \tilde{v}_x = \sum_{y=x-1}^{x+1} \tilde{Z}_{y,x} \tilde{v}_y$$

Bispectrality

Recall $w_n(x) = \langle \tilde{v}_x^* | v_n \rangle$, with $\langle \tilde{v}_x^* | \tilde{V} = \tilde{\lambda}_x \langle \tilde{v}_x^* |$.

Read the matrix element $\langle \tilde{v}_x^* | \tilde{V} Z | v_n \rangle$ *two ways*:

action of $\tilde{V} Z$ on $|v_n\rangle$ (tridiagonal)

$$\sum_{k=n-1}^{n+1} X_{k,n} w_k(x)$$

action of $\tilde{V}$ on $\langle \tilde{v}_x^* |$

$$\tilde{\lambda}_x \sum_{k=n-1}^{n+1} Z_{k,n} w_k(x)$$

Equating them is the **recurrence GEVP** in n :

$$\sum_k X_{k,n} w_k(x) = \tilde{\lambda}_x \sum_k Z_{k,n} w_k(x), \quad X_{k,n} = \langle v_k^* | \tilde{V} Z | v_n \rangle, \quad Z_{k,n} = \langle v_k^* | Z | v_n \rangle.$$

The **difference GEVP** in x is the *same* computation on $\langle \tilde{v}_x^* | Z V | v_n \rangle$ (now using $V | v_n \rangle = \lambda_n | v_n \rangle$)

— so the $w_n(x)$ are **bispectral**.

Biorthogonality

Alongside $w_n(x) = \langle \tilde{v}_x^* | v_n \rangle$ there is a **partner family**, the overlaps of the dual bases:

$$\tilde{w}_n(x) = \langle v_n^* | \tilde{v}_x \rangle.$$

Completeness of the $\tilde{v}_x$ basis, $\sum_x |\tilde{v}_x\rangle \langle \tilde{v}_x^*| = \mathbb{I}$, gives

$$\delta_{mn} = \langle v_n^* | v_m \rangle = \sum_x \langle v_n^* | \tilde{v}_x \rangle \langle \tilde{v}_x^* | v_m \rangle = \sum_x \tilde{w}_n(x) w_m(x),$$

and dually $\sum_n w_n(x) \tilde{w}_n(y) = \delta_{xy}$.

Irreducible Leonard trios

Classifying *all* trios seemed immediately out of reach; the **irreducible** case is the tractable one.

Definition (Crampé, Groenevelt, Labriet, Morey, V, Wagenaar)

An **irreducible Leonard trio** is a trio in which Z is irreducible tridiagonal in both bases and, in addition, there is a basis where Z is diagonal and $V, \tilde{V}$ are *both* irreducible tridiagonal.

In that third basis, (V, Z) and $(\tilde{V}, Z)$ are each a **Leonard pair**, sharing the *same* Z :



each pair a node of the $(q-)$ Askey scheme — the rational function is assembled from *both*.

The bridge: $\tilde{V}Z$ and ZV are algebraic Heun operators

$\tilde{V}Z$ is tridiagonal in *both* eigenbases of the pair (V, Z) : in the V -eigenbasis by the trio definition; in the Z -eigenbasis because Z is diagonal and $\tilde{V}$ tridiagonal there.

The most general operator tridiagonal in both eigenbases of a Leonard pair is its **algebraic Heun operator**, so (likewise ZV for $(\tilde{V}, Z)$):

$$\tilde{V}Z = h_0\mathbb{I} + h_1Z + h_2V + h_3VZ + h_4ZV, \quad ZV = h_5\mathbb{I} + h_6Z + h_7\tilde{V} + h_8\tilde{V}Z + h_9Z\tilde{V}.$$

Conversely, two Leonard pairs sharing Z whose $\tilde{V}Z, ZV$ are these Heun operators form an **irreducible trio**.

The most general trio: two q -Racah pairs sharing Z

Take both (V, Z) and $(\tilde{V}, Z)$ of q -Racah type — the top finite bispectral pairs — sharing one common Z ; being a trio requires $\tilde{V}Z, ZV$ to be their algebraic Heun operators.

- sharing Z ties the parameters ($\tilde{\beta} = \alpha\beta/\tilde{\alpha}$, $\tilde{\gamma} = \gamma$); one constant σ survives (a trio scales Z , but cannot shift it);
- acting on the shared Z -basis, the Heun condition becomes an equality of two *quartics*, forcing equal roots;
- assuming non-degeneracy (each w_n a genuine n -pole function), a unique one-parameter family remains.

The Wilson function as overlap

The overlap $w_n(x) = \langle \tilde{v}_x^* | v_n \rangle$ of this unique trio is the **Wilson biorthogonal rational function** — the balanced, very-well-poised $_{10}\phi_9$ (Wilson, 1991):

$$w_n(x) = {}_{10}\phi_9 \left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-n}, bq^{n+1}, q^{-x}, cq^{x+1}, ad, ae, af \\ \sqrt{a}, -\sqrt{a}, aq^{-n}/b, aq^{n+1}, aq^{-x}/c, aq^{x+1}, q/d, q/e, q/f \end{matrix} ; q, q \right)$$

($bcdef = 1$, $d = q^{N+1}$.)

- known for thirty years, it is *here* interpreted as an **overlap** — and thus shown to be **bispectral**;
- **Wilson** is to the **irreducible trio** what **q -Racah** is to the **Leonard pair**.

A by-product: Wilson as two q -Racah

Inserting the shared Z -basis writes the *same* overlap as a **convolution of two ${}_4\phi_3$** — two q -Racah polynomials:

$$w_n(x) = M \Omega_x \sum_{i=0}^N \Omega_i R_i^{(qR)}(x; \alpha, \beta, \gamma, \frac{\alpha}{\beta\delta}) R_i^{(qR)}(n; \alpha, \beta, \gamma, \delta).$$

In the polynomial limit ($s \rightarrow 0$) this collapses to the **Racah relation** among three q -Racah polynomials.

The trio algebra: a standing problem

The thread of the talk was *algebras*: Leonard pairs lead to the Askey–Wilson algebra, which encodes the bispectral properties of the polynomials.

The Leonard trio and the bispectral rational functions ask the question:

does $(V, \tilde{V}, Z)$ close into a trio algebra — the rational analogue of Askey–Wilson?

- for the Hahn case it has been obtained explicitly — a first, complete entry;
- the general trio algebra, and the classification it would bring, is open — current work.

We expect the eventual trio algebra to have, like the Askey–Wilson algebra, many applications — some of which I had hoped to present.

Bispectral functions lead to rich algebraic structures.
and vice-versa.

- **Orthogonal polynomials:** the theory of **Leonard pairs** — anchored in combinatorics (association schemes) — leads to the **Askey–Wilson algebra**, with higher-rank extensions.
- **Rational functions:** the **Leonard trio** is the analogue of the Leonard pair; its algebraic counterpart, the **trio algebra**, is under development.

These algebras tend to take a life of their own and to bring the associated bispectral functions along.

Joint work with N. Crampé, V. X. Genest, W. Groenevelt, Q. Labriet, L. Morey, S. Tsujimoto, C. Wagenaar,
A. Zhedanov.

Thank you!

Algebras and Bispectrality — Lattice Path Combinatorics, Vienna 2026

Appendix: two definitions

Multivariate (m -variate) P -polynomial association scheme

The classes are indexed by $\vec{n} \in D \subset \mathbb{N}^m$, and every adjacency matrix is a *multivariate* polynomial of multidegree $\vec{n}$ in m generators, $A_{\vec{n}} = v_{\vec{n}}(A_{e_1}, \dots, A_{e_m})$, compatible with a fixed monomial order $\leq$ on $\mathbb{N}^m$. The $v_{\vec{n}}$ are **multivariate (Tratnik) orthogonal polynomials**. ($m = 1$: the classical $A_i = v_i(A_1)$.)

m -distance-regular graph

A connected graph with an m -partition of its edges (m colours) and a vector m -distance $d_m(x, y) \in \mathbb{N}^m$ (relative to a monomial order): $e_1, \dots, e_m$ are realized m -distances, and the number p_{ab}^c of vertices z at m -distance a from x and b from y depends only on $c = d_m(x, y)$. ($m = 1$: the usual distance-regular graph.)

Correspondence: m -distance-regular graphs $\iff m$ -variate P -polynomial schemes.