# Crossover from subcritical to critical decay: random walk, self-avoiding walk, percolation

Gordon Slade, University of British Columbia

joint work with Yucheng Liu, Peking University

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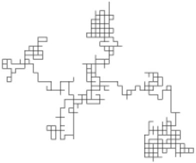

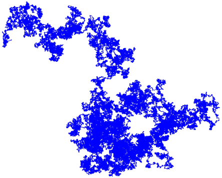

## Lattice models and their two-point functions

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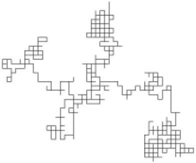
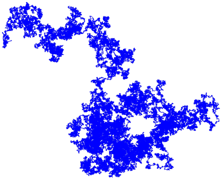
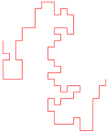
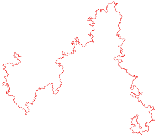




Self-avoiding walk: $G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n$, $z_c = 1/\lim_{n\to\infty} c_n^{1/n} \in (0,\infty)$





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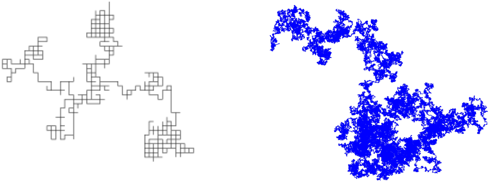

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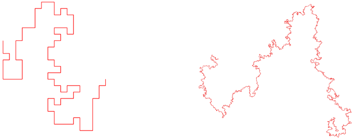
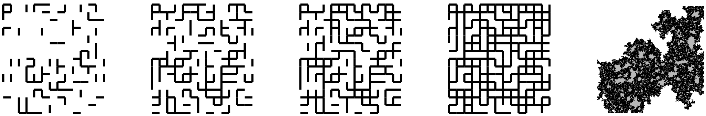


Percolation: $G_z(x) = \mathbb{P}_z(0 \leftrightarrow x)$, $z_c \in (0, 1)$



# Convolution equation

In all three examples, $G_z$ satisfies a convolution equation

$$G_z = h_z + J_z * G_z, \qquad (f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(y) g(x - y).$$

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Random walk: With $h_z = \delta_0$, $J_z = zD$, and $D$ the transition probability kernel $D(x) \geq 0$, $\sum_{x \in \mathbb{Z}^d} D(x) = 1$ (Markov property).

Self-avoiding walk for $d > 4$: With $h_z = \delta_0$ and $J_z = 2dzP + \Pi_z$, for $P(x) = \frac{1}{2d} \mathbb{1}\{\|x\|_1 = 1\}$ and $\Pi_z$ from lace expansion.

Percolation for $d > 6$: With $h_z = \delta_0 + \Pi_z$ from lace expansion, and $J_z = 2dzP * h_z$.

More examples: Ising model, $|\varphi|^4$ model, lattice trees, lattice animals.

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Formal solution to $(\delta_0 - J_z) * G_z = h_z$: First solve for the *Green function* $G_z^o = (\delta_0 - J_z)^{-1}\delta_0$. Then

$$G_z = G_z^o * h_z.$$

For random walk, $G_z = G_z^o$ is called the lattice Green function, and when $D = P$, $G_z$ is the inverse Laplace operator (massive for $z < 1$). For self-avoiding walk and percolation, $J_z$ is *infinite-range* and can take *negative values*.

# Our goals

1) We want a *general* theorem, which derives the long-distance asymptotic decay of $G_z$ obeying $G_z = h_z + J_z * G_z$ from *checkable* hypotheses on $h_z, J_z$.

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Over the past 40 years, the decay has been proved for several statistical-mechanical models for *fixed* $z < z_c$ (Campanino, Chayes, Chayes, Ioffe, Velenik, ... via renewal methods) and also for $z = z_c$ (Bolthausen, Brydges, Hara, van der Hofstad, Liu, Slade, ... via lace expansion and deconvolution).

The second goal has been achieved for simple random walk ($D = P$, Michta–Slade 2022), $2$-$d$ Ising (exact solution) and partially achieved for the $2$-$d$ random cluster model with $1 \leq q < 4$ (D'Alimonte–Manolescu 2026).

Beginning with the work of Ornstein and Zernike (1914, 1916), it has been understood that the subcritical two-point function typically obeys the OZ decay

$$G_z(x) \sim c_{\hat{x},z} \frac{1}{|x|_z^{(d-1)/2}} e^{-m_z |x|_z} \quad \text{as } x \to \infty \text{ with } z < z_c \text{ fixed}$$

with $m_z > 0$ (the *inverse correlation length*), an anisotropic norm satisfying $\|x\|_\infty \leq |x|_z \leq \|x\|_1$, and a $z$-dependent and direction-dependent constant $c_{\hat{x},z}$.

# Ornstein-Zernike vs critical decay

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For $z = z_c$, in the mean-field regime ($d > 2$ for RW, $d > 4$ for SAW, $d > 6$ for percolation), the decay is

$$G_{z_c}(x) \sim c \frac{1}{\|x\|_2^{d-2}}.$$

At the critical point we have $m_{z_c} = 0$, but the powers $|x|_z^{(d-1)/2}$ and $\|x\|_2^{d-2}$ do not match (except accidentally for $d = 3$).

We want a theorem that says that if $h_z, J_z$ are nice, then $G_z$ obeys an asymptotic formula that encompasses both the OZ and critical decay.

# Exponential tilt and the Wulff shape

Given $\mu \in \mathbb{R}^d$, the *exponential tilt* of $f : \mathbb{Z}^d \to \mathbb{R}$ is

$$f^{(\mu)}(x) = f(x)e^{\mu \cdot x}.$$

Let $S : \mathbb{Z}^d \to [0, \infty)$ be $\mathbb{Z}^d$-symmetric. The *tilted susceptibility* is

$$\chi^{(\mu)} = \sum_{x \in \mathbb{Z}^d} S^{(\mu)}(x) = \sum_{x \in \mathbb{Z}^d} S(x) \cosh(\mu \cdot x).$$

The *Wulff shape* $\overline{\Omega}$ is the closure of the set

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Let

$$m_S = \sup\{t \geq 0 : te_1 \in \Omega\}.$$

**Lemma 1.** Under Assumption I, $\Omega$ is bounded and convex, $m_S \in (0, \mu_1]$, and

$$\{\mu \in \mathbb{R}^d : \|\mu\|_1 \leq m_S\} \subset \overline{\Omega} \subset \{\mu \in \mathbb{R}^d : \|\mu\|_\infty \leq m_S\},$$

and if $\mu \in \partial\Omega$ then $\|\mu\|_\infty \leq m_S \leq \|\mu\|_1$ and $\chi^{(\mu)} = \infty$.

**Assumption II.** For $d \geq 1$, $S$ obeys $S = \delta_0 + J * S$ $(h = \delta_0$ for simplicity$)$ with $\mathbb{Z}^d$-symmetric $J \in L^1(\mathbb{Z}^d)$, and $\exists M, K_{\mathrm{IR}}, \zeta$ such that $J^{(\mu)} \in \mathcal{Q}_{M,K_{\mathrm{IR}},\zeta}$ uniformly in $\mu \in \overline{\Omega}$.

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{-ik \cdot x}$$

**Definition.** Let $d \geq 1$, $M, K_{\mathrm{IR}} > 0$, $\zeta \in (0, 2]$. The set $\mathcal{Q}_{M,K_{\mathrm{IR}},\zeta}$ consists of $Q : \mathbb{Z}^d \to \mathbb{R}$ such that

$$|Q(0)| + \left\| |y|^{2+\zeta} Q(y) \right\|_1 + \mathbb{1}_{d \geq 4} \left\| |y|^{d-1} Q(y) \right\|_{\frac{d}{3} \wedge 2} \leq M,$$
$$\mathrm{Re}[\hat{Q}(0) - \hat{Q}(k)] \geq K_{\mathrm{IR}} |k|^2 \qquad (k \in [-\pi, \pi]^d).$$

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For such a $Q$, we define

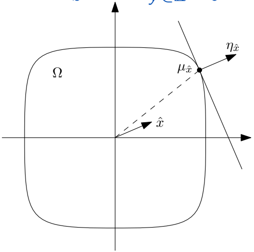
$$\eta_j = \sum_{y \in \mathbb{Z}^d} y_j Q(y), \qquad \Lambda_{jl} = \sum_{y \in \mathbb{Z}^d} y_j y_l Q(y).$$

Easy: $|\eta_j| \leq M, \quad |\Lambda_{jl}| \leq M, \quad k \cdot \Lambda k \asymp |k|^2 \asymp k \cdot \Lambda^{-1} k$.

Note that $\chi^{(\mu)} = 1 + (\sum_{y \in \mathbb{Z}^d} J^{(\mu)}(y)) \chi^{(\mu)}$.

Suppose that Assumptions I and II hold. Then

- $\overline{\Omega}$ is strictly convex ,
- $\forall \mu \in \overline{\Omega}$, $\sum_{y \in \mathbb{Z}^d} J^{(\mu)}(y) \leq 1$ with equality if and only if $\mu \in \partial\Omega$,
- $\forall x \neq 0 \; \exists! \mu_{\hat{x}} \in \partial\Omega$ such that $\mu_{\hat{x}} \cdot x = \max_{\mu \in \overline{\Omega}} \mu \cdot x$,
- the vector $\eta_{\hat{x}}$ with components $\eta_{\hat{x},j} = \sum_{y \in \mathbb{Z}^d} y_j J^{(\mu_{\hat{x}})}(y)$ points in the same direction as $x$,



# Geometry of the Wulff shape

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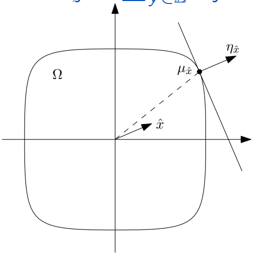


- The function $|\cdot|_S : \mathbb{R}^d \to [0, \infty)$ defined by $|0|_S = 0$ and

$$|x|_S = \frac{\mu_{\hat{x}} \cdot x}{m_S} \qquad (x \neq 0)$$

  is a norm with $\|x\|_\infty \leq |x|_S \leq \|x\|_1 \; \forall x \in \mathbb{R}^d$.
  (In fact, $\overline{\Omega}$ is the ball of radius $m_S$ for the dual norm.)

The *heat kernel* of Brownian motion on $\mathbb{R}^d$ with drift $\eta$ and covariance $\Lambda$ is

$$\rho_t(x;\eta,\Lambda) = \frac{1}{\sqrt{\det\Lambda}}\frac{1}{(2\pi t)^{d/2}}\exp\Big\{-\frac{1}{2t}(x-t\eta)\cdot\Lambda^{-1}(x-t\eta)\Big\}.$$

Its *Green function* is the integral

$$\mathbb{C}(x;\eta,\Lambda) = \int_0^\infty \mathrm{d}t\,\rho_t(x;\eta,\Lambda) \qquad (x\in\mathbb{R}^d).$$

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**Crossover Theorem.** Let $d \geq 1$. Suppose Assumptions I and II hold. For $d > 2$,

$$S(x) = \mathbb{C}(x;\eta_{\hat{x}},\Lambda_{\hat{x}})e^{-m_S|x|_S}\Big[1 + O\Big(\frac{1}{|x|^{\varepsilon}}\Big)\Big]$$

with the constant depending only on $d, \zeta, M, K_{\mathrm{IR}}$. For $d \leq 2$, we also require $m_S|x| \geq s_0$ for some $s_0 > 0$.

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Complete information about $\mathbb{C}$ is available, e.g.,

$$\mathbb{C}(x;\|\eta\|_2\hat{x},\Lambda) = \frac{1}{\|\eta\|_2} \qquad (d=1)$$

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It does not decay exponentially when the drift is aligned with $x$ (our case).

# Crossover from OZ to critical decay

**Corollary.** Let $d > 2$ and suppose that Assumptions I and II hold. There exists $R > 0$ such that for all $|x|_S \geq R$,

$$S(x) \asymp \frac{\max\{1, m_S|x|_S\}^{(d-3)/2}}{|x|_S^{d-2}} e^{-m_S|x|_S}$$

with constants that depend only on $d, \zeta, M, K_{\mathrm{IR}}$.

This provides the crossover: For $m_S|x|_S \geq 1$ it gives OZ decay

$$S(x) \asymp \frac{m_S^{(d-3)/2}}{|x|_S^{(d-1)/2}} e^{-m_S|x|_S}$$

and for $m_S|x|_S \leq 1$ it gives critical decay

$$S(x) \asymp \frac{1}{|x|_S^{d-2}}.$$

(The Crossover Theorem says much more than this.)

# Idea of proof of Crossover Theorem

Tilting of $S = \delta_0 + J * S$ by $\mu_{\hat{x}}$ gives

$$S^{(\mu_{\hat{x}})}(x) = \delta_{0,x} + (J^{(\mu_{\hat{x}})} * S^{(\mu_{\hat{x}})})(x),$$

and the inverse Fourier transform then gives

$$S^{(\mu_{\hat{x}})}(x) = \int_{[-\pi,\pi]^d} \frac{e^{ik\cdot x}}{1 - \hat{J}^{(\mu_{\hat{x}})}(k)} \frac{\mathrm{d}k}{(2\pi)^d} = \int_0^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} e^{ik\cdot x} e^{-t[1-\hat{J}^{(\mu_{\hat{x}})}(k)]}.$$

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Idea: large $x$ behaviour is dominated by small $k$. Taylor expand

$$1 - \hat{J}^{(\mu_{\hat{x}})}(k) \approx ik \cdot \eta_{\hat{x}} + \frac{1}{2} k \cdot \Lambda_{\hat{x}} k$$

and replace the integral over $[-\pi,\pi]^d$ by an integral over $\mathbb{R}^d$. This gives

$$\begin{aligned} S^{(\mu_{\hat{x}})}(x) &\approx \int_0^\infty \mathrm{d}t \int_{\mathbb{R}^d} \frac{\mathrm{d}k}{(2\pi)^d} e^{ik\cdot x} e^{-t[ik\cdot\eta_{\hat{x}} + \frac{1}{2}k\cdot\Lambda_{\hat{x}}k]} \\ &= \int_0^\infty \mathrm{d}t \rho_t(x; \eta_{\hat{x}}, \Lambda_{\hat{x}}) = \mathbb{C}(x; \eta_{\hat{x}}, \Lambda_{\hat{x}}). \end{aligned}$$

The proof requires controlling the approximation errors. A computation like this was done by Hara (2008) in the zero drift case. We extend it to nonzero drift.

Now suppose that

$$S_z = \delta_0 + J_z * S_z \qquad z \in [z_c - \delta, z_c),$$

that Assumptions I and II hold uniformly in $\mu \in \overline{\Omega}_z$ and in $z$, and that $\lim_{z \to z_c} m_z = 0$. Let $\sigma_z^2 = \sum_{x \in \mathbb{Z}^d} \|x\|_2^2 J_z(x)$. Then as $z \to z_c$:

- Isotropic limit:

$$|x|_z = \|x\|_2 [1 + O(m_z^{\zeta \wedge 2})].$$

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- For $\phi > 0$, the correlation lengths $\xi_\phi(z) = [\frac{1}{\chi(z)} \sum_{x \in \mathbb{Z}^d} \|x\|_2^\phi S_z(x)]^{1/\phi}$ have universal ratios (independent of $J_z$)

$$\xi_\phi(z) \sim 2 \left( \frac{\Gamma(\frac{\phi+2}{2}) \Gamma(\frac{\phi+d}{2})}{\Gamma(\frac{d}{2})} \right)^{1/\phi} \frac{1}{m_z}.$$

Thus, all correlation lengths are asymptotically related by universal constants.

# Application 1: Random walk

Consider a RW $(X_n)_{n\geq 0}$ with $X_0 = 0$, $\mathbb{P}(X_1 = x) = D(x)$ where $D$ is $\mathbb{Z}^d$-symmetric, not supported on a proper sublattice, with $D(x) = o(e^{-a\|x\|_2})$ for every $a > 0$. Its variance $\sigma_D^2 = \sum_{x\in\mathbb{Z}^d} \|x\|_2^2 D(x)$ is finite. For $z \in (0, 1)$, its Green function

$$S_z(x) = \sum_{n=0}^{\infty} \mathbb{P}(X_n = x) z^n$$

satisfies

$$S_z = \delta_0 + J_z * S_z \quad \text{with} \quad J_z = zD.$$

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**Theorem.** If $D$ has finite support $\mathcal{S}$ then $\lim_{z\to 0} |x|_z$ defines a norm whose unit ball is

$$\{x : |x|_0 = 1\} = \frac{\mathrm{Conv}(\mathcal{S})}{\mathrm{rad}(\mathcal{S})}.$$

Explicit computations of the norm can be done via Lagrange mulitpliers, which leads to solving

$$z\hat{D}^{(\mu)}(0) = 1, \qquad z\nabla_\mu \hat{D}^{(\mu)}(0) = \lambda x.$$

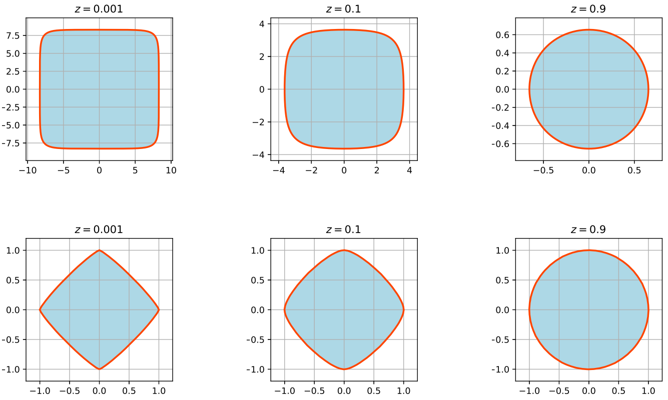
For simple random walk, i.e., $D(x) = \frac{1}{2d}\mathbb{1}\{\|x\|_1 = 1\}$, this system has solution

$$\frac{1}{d}\sum_{j=1}^{d}\sqrt{1+\Big(\frac{d\lambda_{z,x}x_j}{z}\Big)^2} = \frac{1}{z}, \qquad |x|_z = \frac{1}{m_z}\sum_{j=1}^{d} x_j \operatorname{arcsinh}\frac{d\lambda_{z,x}x_j}{z}.$$

By calculus, the norm interpolates monotonically between $|x|_0 = \|x\|_1$ and $|x|_1 = \|x\|_2$.

# Application 1: Norm for $D(x) = \frac{1}{2d}\mathbb{1}\{\|x\|_1 = 1\}$

Wulff shape (upper) and unit ball for norm (lower) for simple random walk:



Identifies a function $\pi_m(x)$ such that for $n \geqslant 1$,

$$c_n(x) = \sum_{y \in \mathbb{Z}^d} c_1(y) c_{n-1}(x-y) + \sum_{m=2}^{n} \sum_{y \in \mathbb{Z}^d} \pi_m(y) c_{n-m}(x-y).$$

# Application 2: Self-avoiding walk – lace expansion

Identifies a function $\pi_m(x)$ such that for $n \geqslant 1$,

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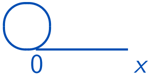
Start with

$$c_n(x) = \sum_{y \in \mathbb{Z}^d} c_1(y) c_{n-1}(x-y) - R_n^{(1)}(x)$$

where

$$R_n^{(1)}(x) =$$



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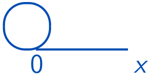



Inclusion-exclusion again:

$$R_n^{(1)}(x) = \sum_{m=2}^{n} u_m c_{n-m}(x) - R_n^{(2)}(x)$$

where

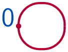
$$R_n^{(2)}(x) =$$

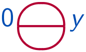
# Application 2: Self-avoiding walk – lace expansion continued

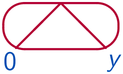
Repetition leads to

$$c_n(x) = (c_1 * c_{n-1})(x) + \sum_{m=2}^{n} (\pi_m * c_{n-m})(x)$$

with

$\pi_m(y) = -\delta_{0,y}$ 

$+$ 

$-$ 

$+ \cdots$

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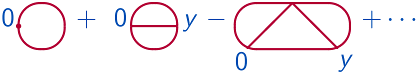



Then

$$G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n = \delta_{0,x} + z(c_1 * G_z)(x) + (\Pi_z * G_z)(x),$$

where

$$\Pi_z(y) = \sum_{n=2}^{\infty} \pi_n(y) z^n.$$

Convolution equation:

$$G_z = \delta_0 + J_z * G_z, \qquad J_z = 2dzP + \Pi_z.$$

For $d \geq 5$, $\Pi_z$ is well controlled (Brydges–Spencer 1985, Hara–Slade 1992).

## Application 2: Self-avoiding walk – crossover

**Theorem.** For $d \geq 5$ and $z \in [z_c - \delta, z_c)$, Assumptions I and II hold for self-avoiding walk, uniformly in $z$.

**Proof.** Apply and extend lace expansion estimates.

**Corollaries.** Sample consequences:

- For $z \in [z_c - \delta, z_c]$ *including* $z_c$,

$$G_z(x) = \mathbb{C}(x; \eta_{\hat{x},z}, \Lambda_{\hat{x},z}) e^{-m_z |x|_z} \left[1 + O\Big(\frac{1}{|x|^{\varepsilon}}\Big)\right],$$

-

$$G_z(x) \asymp \frac{\max\{1, m_z |x|_z\}^{(d-3)/2}}{|x|_z^{d-2}} e^{-m_z |x|_z},$$

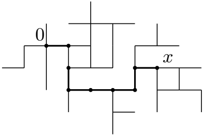
- as $z \to z_c$, $|x|_z \sim \|x\|_2$ and $(\sigma_{z_c}^2 = \sum_{y \in \mathbb{Z}^d} \|y\|_2^2 J_{z_c}(y))$

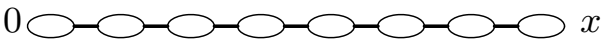
$$m_z^2 \sim \frac{2d\partial_z \hat{J}_{z_c}(0)}{\sigma_{z_c}^2}(1 - z/z_c)^1, \qquad \xi_\phi(z) \sim 2\bigg(\frac{\Gamma(\frac{\phi+2}{2})\Gamma(\frac{\phi+d}{2})}{\Gamma(\frac{d}{2})}\bigg)^{1/\phi} \frac{1}{m_z}.$$

# Application 3: Percolation lace expansion

Let

$$\tau_p(x) = \mathbb{P}_p(0 \longleftrightarrow x).$$




The lace expansion for percolation provides a convolution equation

$$\tau_p = h_p + J_p * \tau_p \quad \text{with} \quad h_p = \delta_0 + \Pi_p, \quad J_p = 2dpP * h_p.$$

# Application 3: Percolation – crossover

The function $\Pi_p$ is well controlled for $d \geq 11$ and, for spread-out models, for all $d > 6$ (Hara–Slade 1990, Fitzner–van der Hofstad 2017). An extension of this control gives:

**Theorem.** For $d \geq 15$ and $p \in [p_c - \delta, p_c)$, Assumptions I and II hold for percolation, uniformly in $p$.
**Corollaries.** Sample consequences:

- For $p \in [p_c - \delta, p_c]$ *including* $p_c$,

$$\tau_p(x) \sim \mathbb{C}(x; \eta_{\hat{x},p}, \Lambda_{\hat{x},p}) e^{-m_p |x|_p} \asymp \frac{\max\{1, m_p |x|_p\}^{(d-3)/2}}{|x|_p^{d-2}} e^{-m_p |x|_p},$$

- as $p \to p_c$, $|x|_p \sim \|x\|_2$ and

$$\chi(p) \sim \frac{2d \hat{h}_{p_c}(0)}{\sigma_{p_c}^2} \frac{1}{m_p^2}, \qquad \xi_\phi(p) \sim 2 \left( \frac{\Gamma(\frac{\phi+2}{2}) \Gamma(\frac{\phi+d}{2})}{\Gamma(\frac{d}{2})} \right)^{1/\phi} \frac{1}{m_p}.$$

- For convolution equations of the form $S = h + J * S$, under our two Assumptions, the long-distance asymptotic behaviour of $S(x)$ is exactly given by the Green function for a Brownian motion whose drift and covariance are determined by a variational problem, times an anisotropic exponential.

# Summary

- For convolution equations of the form $S = h + J * S$, under our two Assumptions, the long-distance asymptotic behaviour of $S(x)$ is exactly given by the Green function for a Brownian motion whose drift and covariance are determined by a variational problem, times an anisotropic exponential.
- When $S_z = h_z + J_z * S_z$ is indexed by $z \in (0, z_c)$, if the Assumptions hold uniformly in $z$, this allows for a complete understanding of crossover from Ornstein–Zernike decay $|x|_z^{-(d-1)/2} e^{-m_z |x|_z}$ to critical decay $\|x\|_2^{-(d-2)}$ as $z \to z_c$.

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- Other consequences follow, including the universal equivalence of all correlation lengths.
- The Assumptions do hold uniformly near $z_c$ for all random walks with super-exponentially decaying transition kernel, for self-avoiding walk in dimensions $d \geq 5$, and for percolation in dimensions $d \geq 15$.

# The end

Thank you!