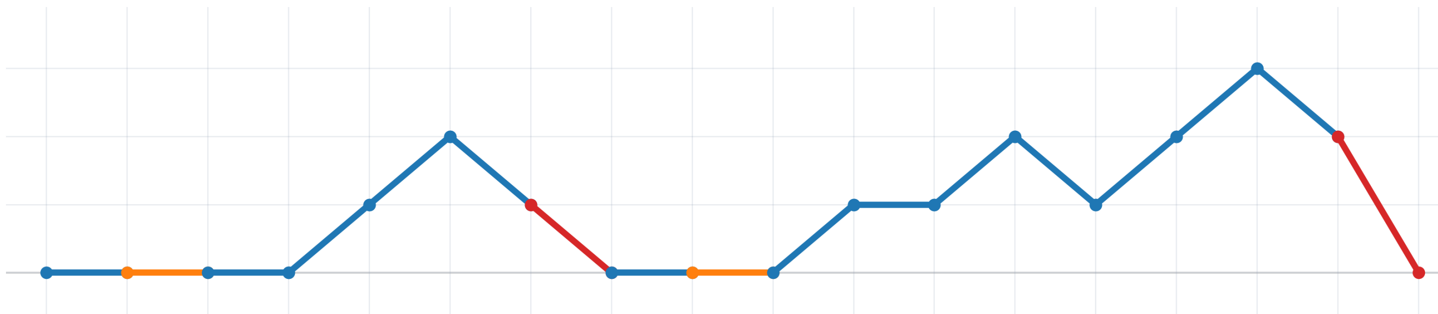
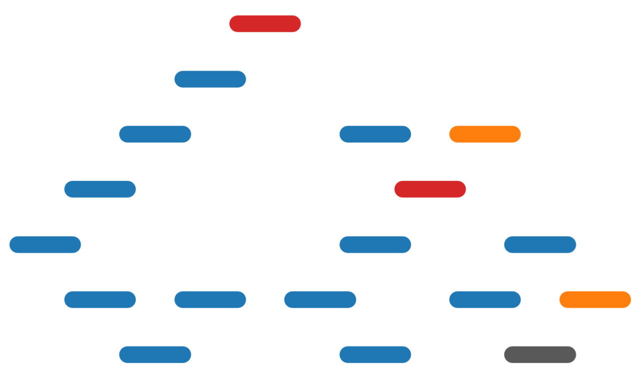
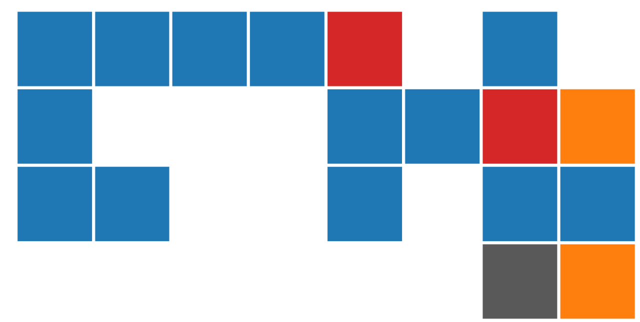
Polyominoes, lattice paths and catastrophes

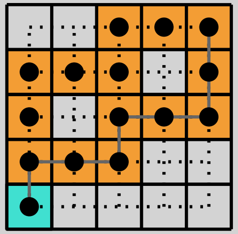
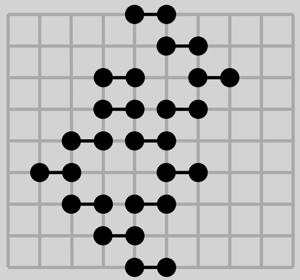






Florian Schager · Michael Wallner

Outline of the talk

Background	lattice paths with catastrophes, heaps of dimers, polyominoes	 

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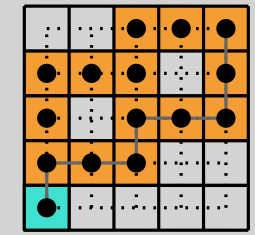
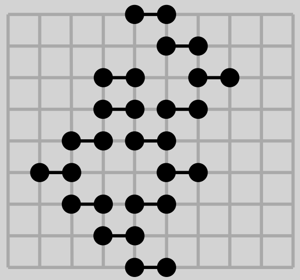


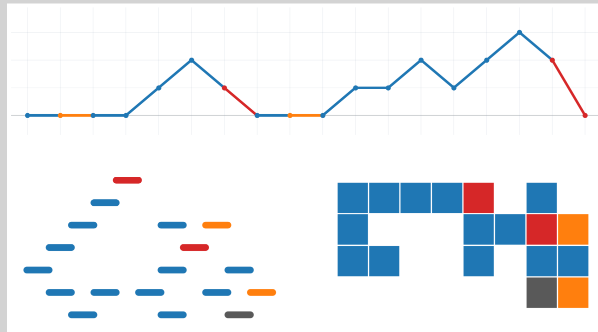


Bijection

Motzkin excursions with catastrophes

↕

stacked directed animals



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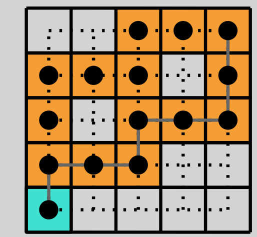

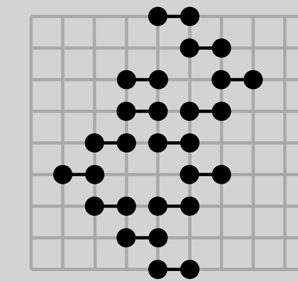

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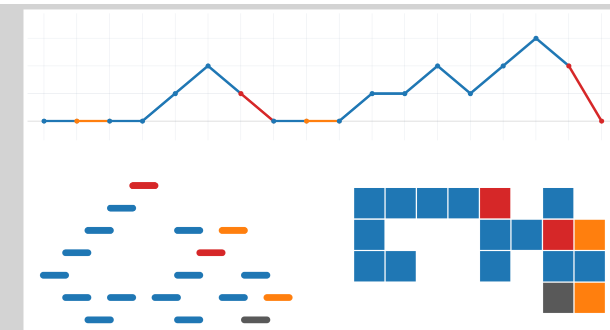
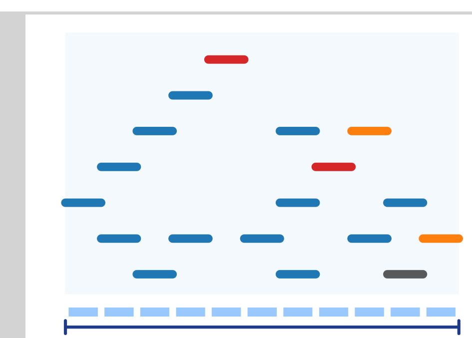


Application

translate lattice path statistics to
the world of heaps



Polyominoes

Polyomino or **Lattice animal**: a connected union of cells on a lattice.

Directed animals

- one **source** cell
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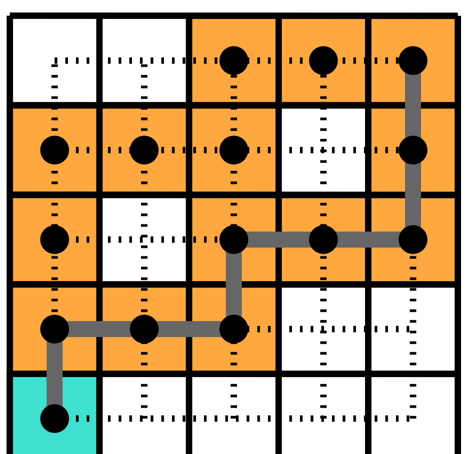
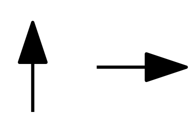


Directions:



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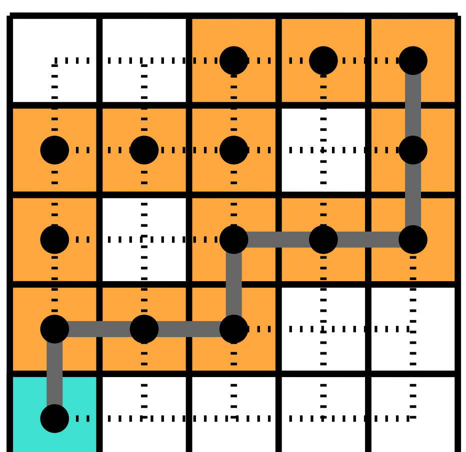
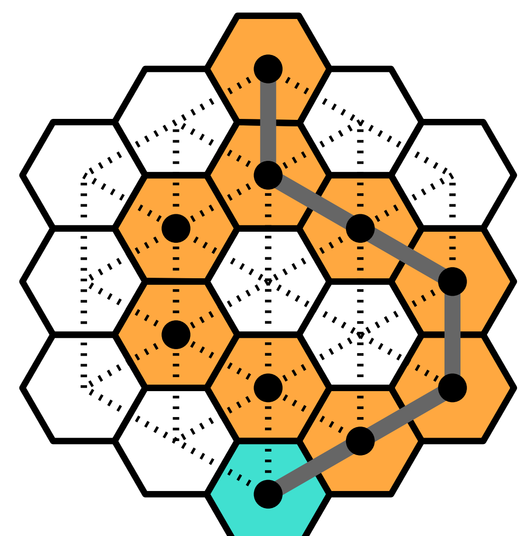


Directions: ↑ →

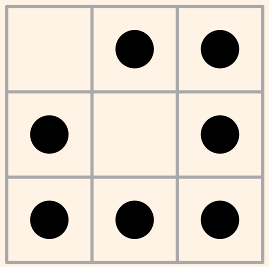
hexagonal lattice

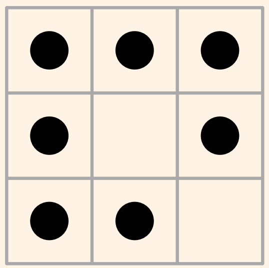


Directions: ↑ ↗ ↖

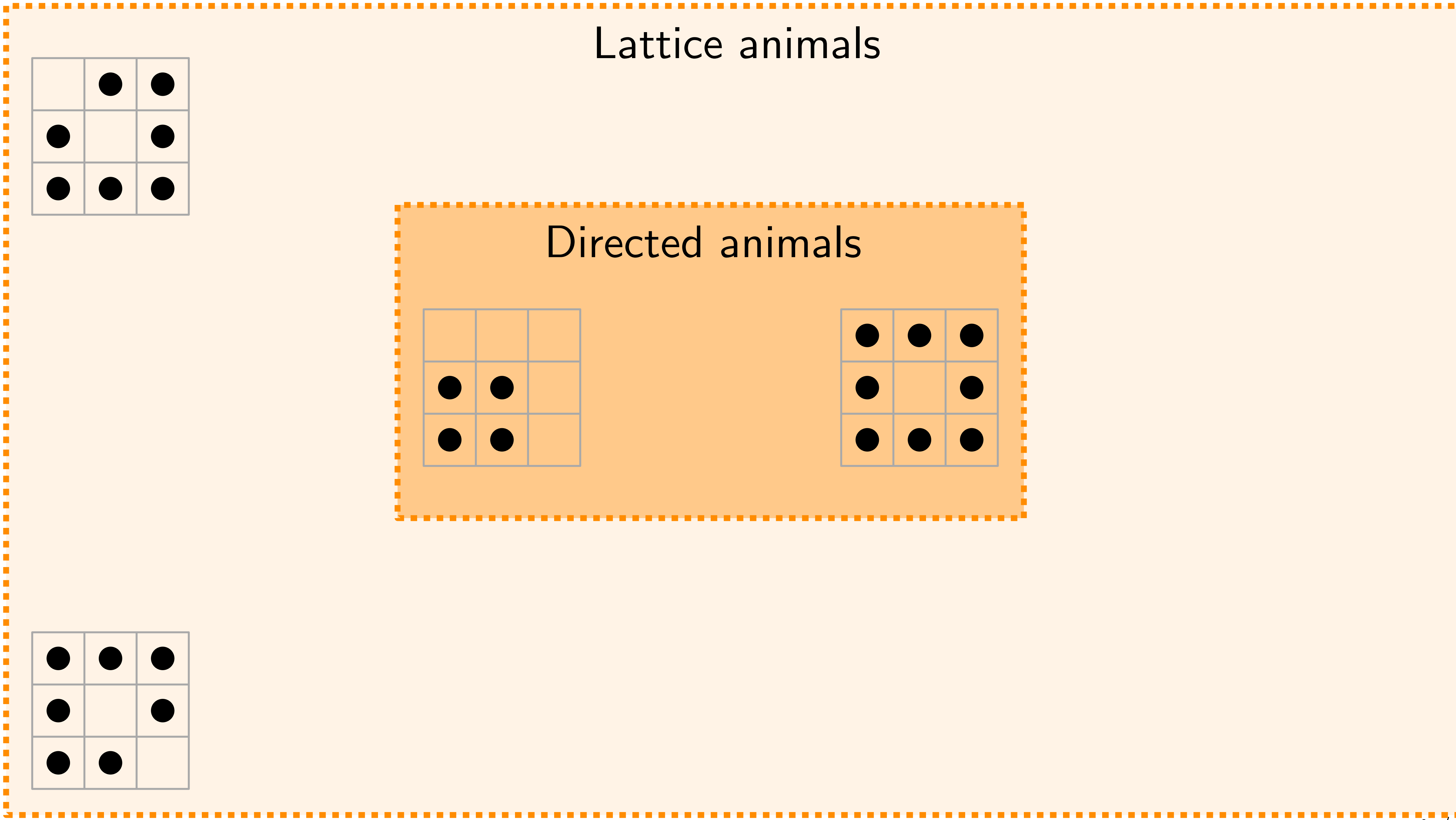
Subclasses of lattice animals

Lattice animals

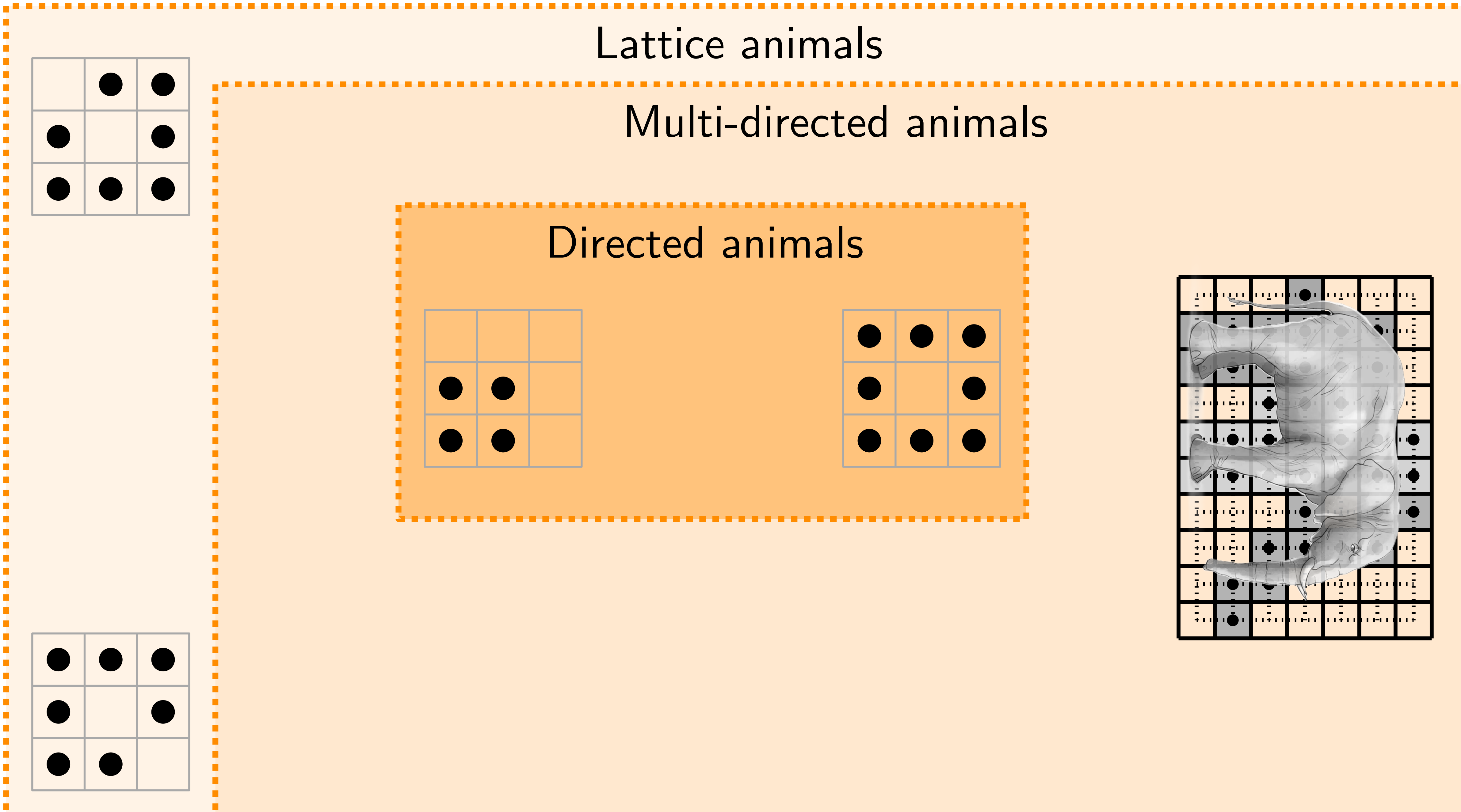




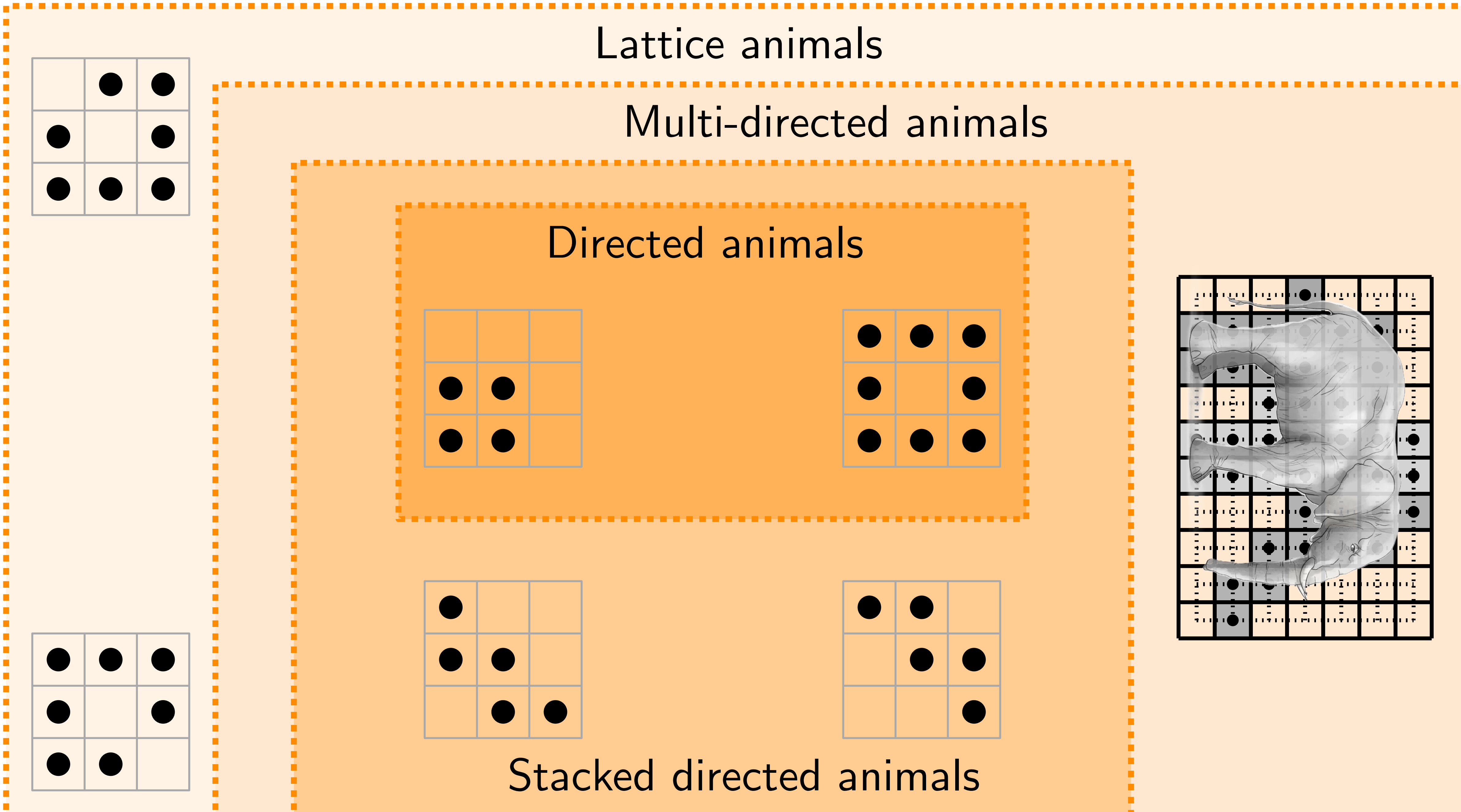
Subclasses of lattice animals



Subclasses of lattice animals

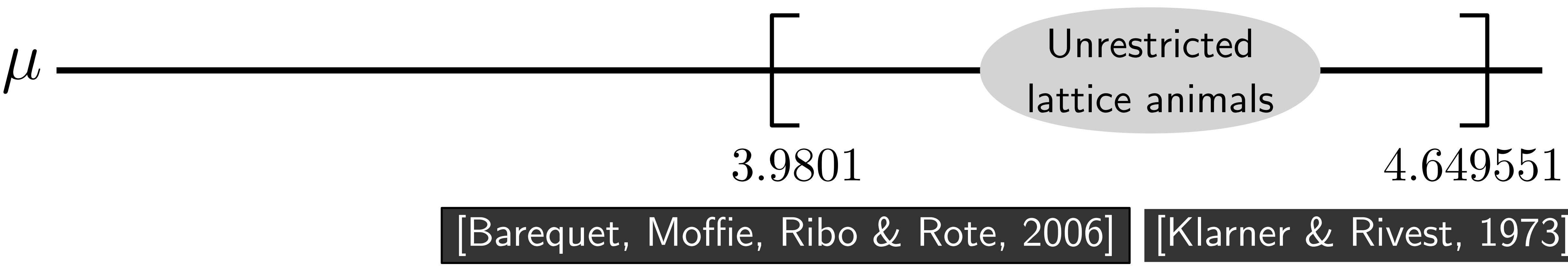


Subclasses of lattice animals



Enumeration of lattice animals

Klarner's growth constant $\mu := \lim_{n\to\infty} a_n^{1/n}$



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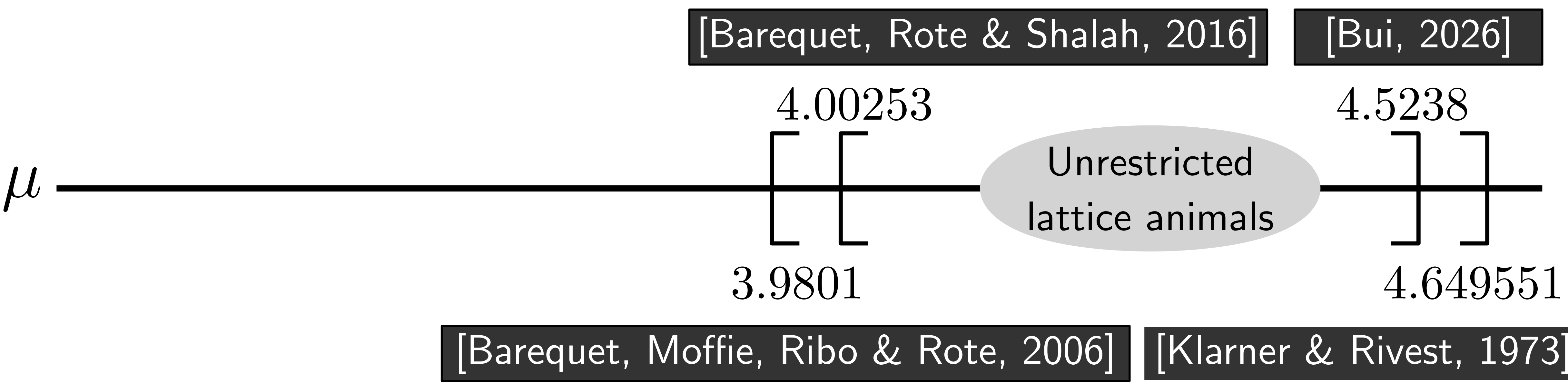

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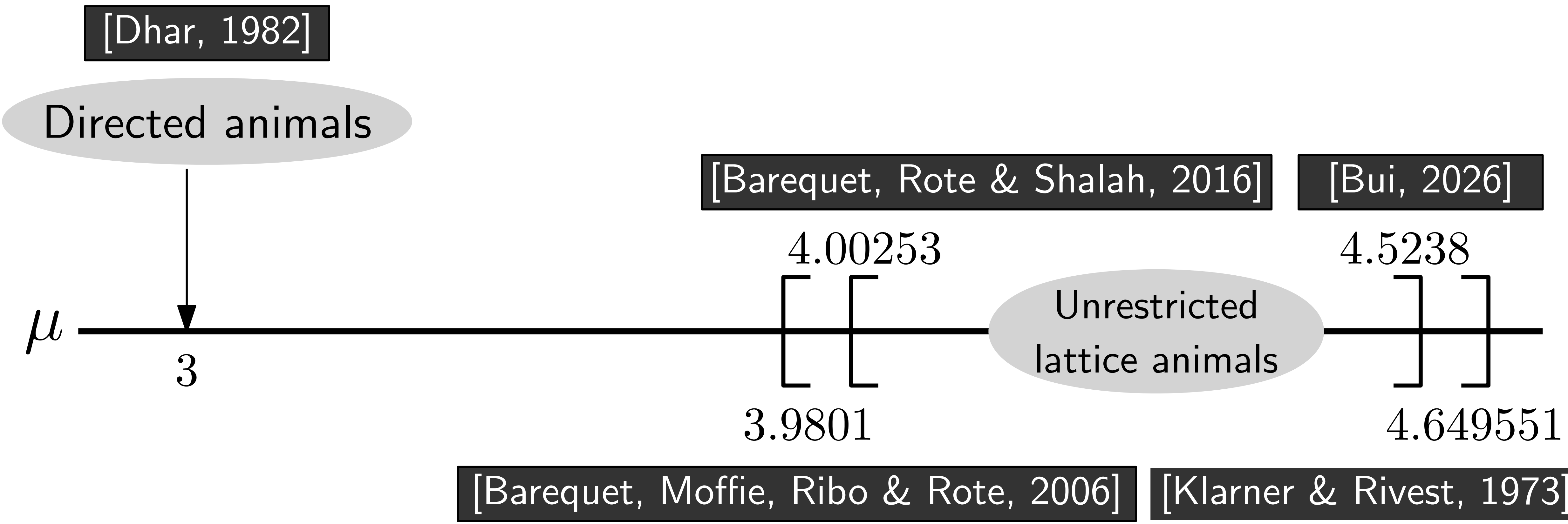

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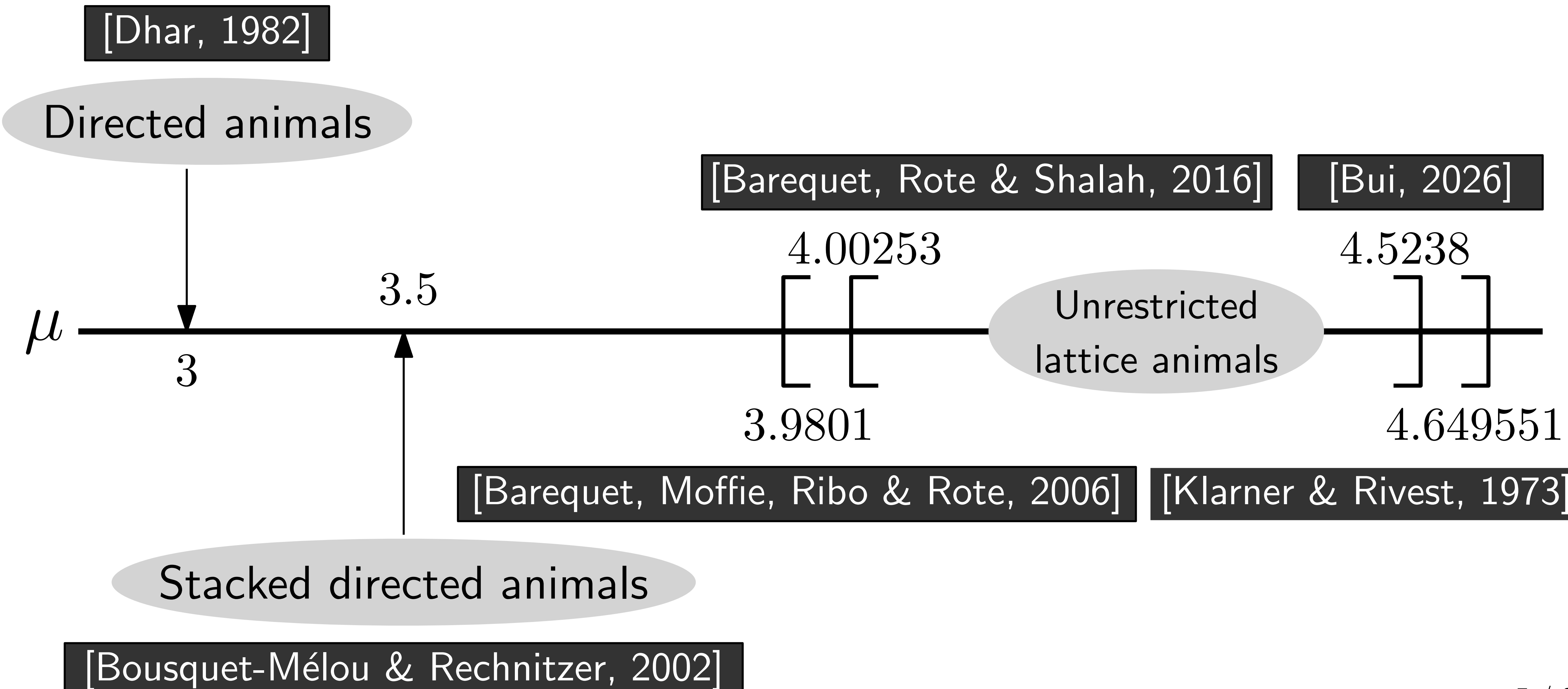
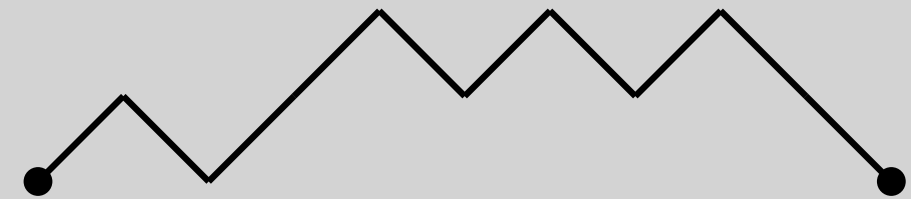
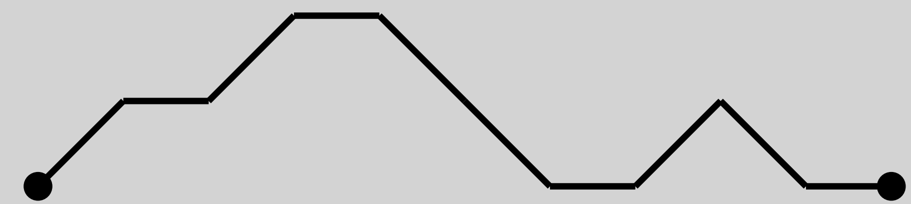


Lattice paths with catastrophes

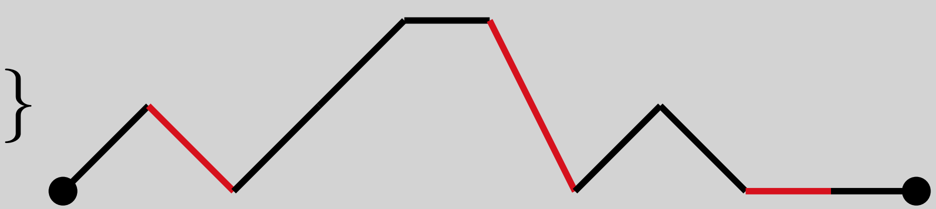
Dyck paths $\mathcal{S} = \{1, -1\}$



Motzkin paths $\mathcal{S} = \{1, 0, -1\}$



Motzkin paths with catastrophes $\mathcal{S} = \{1, 0, -1, \mathbf{C}\}$



Catastrophe: A step, allowed at any height, that takes the path down to the x-axis

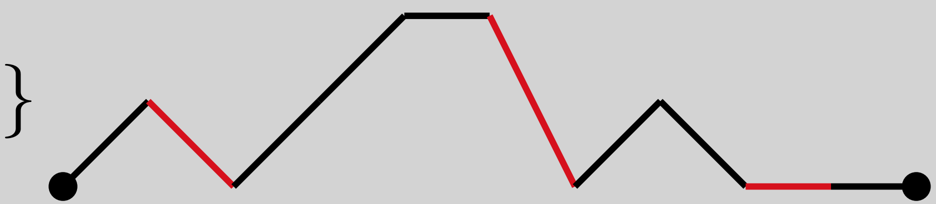
[Banderier & Wallner, 2017]

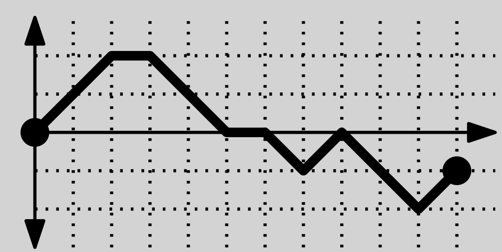
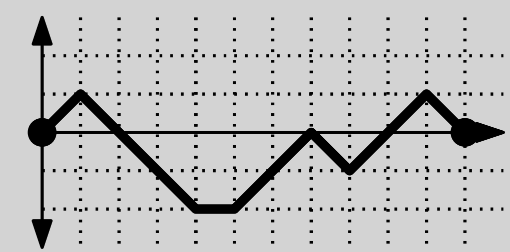
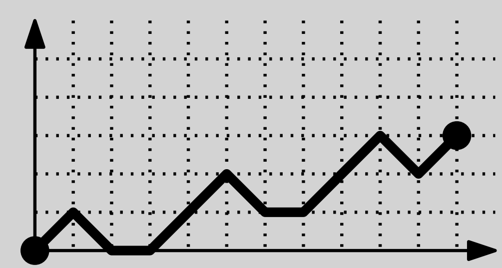
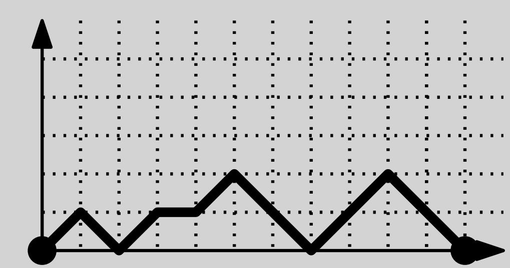
Lattice paths with catastrophes

Motzkin paths

with catastrophes

$\mathcal{S} = \{1, 0, -1, \textbf{C}\}$



Constraints:	ending anywhere	ending at zero
unconstrained (on $\mathbb{Z}$)	 walk or path	 bridge
constrained (on $\mathbb{Z}_+$)	 meander	 excursion

Generating functions of lattice paths with catastrophes

Theorem [Banderier & Wallner, 2017]

The generating function $C(z,u)$ of meanders with catastrophes is algebraic and satisfies

$$C(z,u) = \frac{1}{1 - z \cdot M(z,1)} \cdot \frac{\prod_{i=1}^{c}(u - u_i(z))}{u^c(1 - zP(u))},$$

where

- $M(z,u)$ is the generating function of classical meanders and
- $u_1, \ldots, u_c$ are the small roots and $v_1, \ldots, v_d$ the large roots of the kernel equation.

In particular, the generating function $E(z)$ of excursions with catastrophes satisfies

$$E(z) = \frac{(-1)^{c-1}}{1 - z \cdot M(z,1)} \cdot \frac{\prod_{i=1}^{c} u_i(z)}{z p_{-c}}$$

Heaps of dimers

Definition [Viennot, 1985]

Drop a finite number of dimers towards the x-axis until each dimer either touches the x-axis or another dimer.

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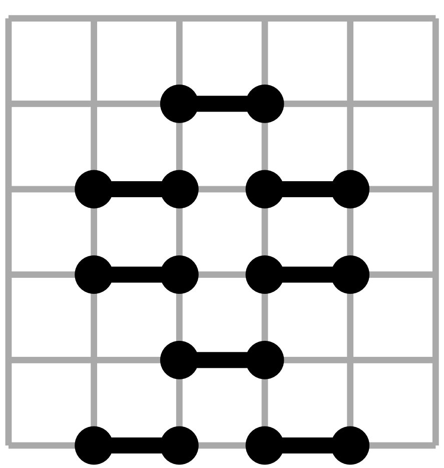
Connected heap



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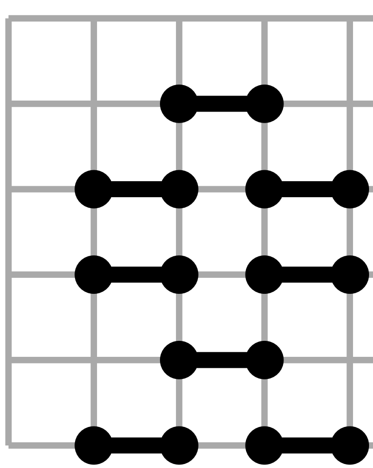

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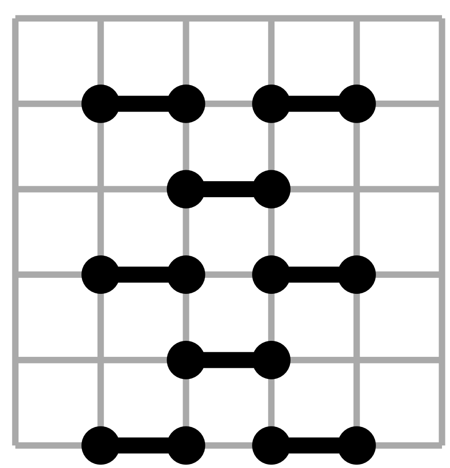
Strict heap



no two dimers on top of each other

Heaps of dimers

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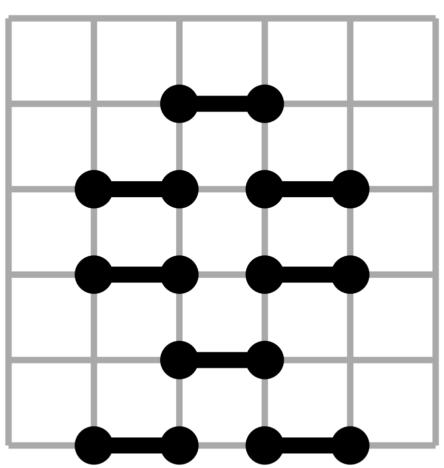
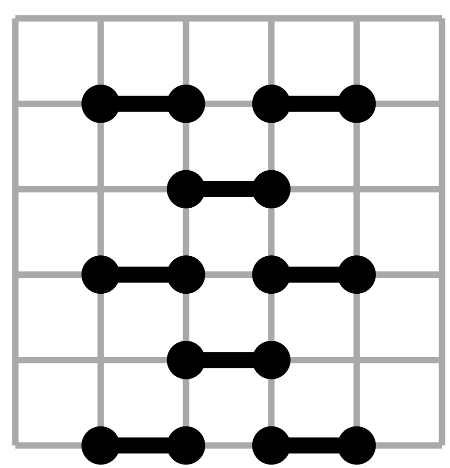
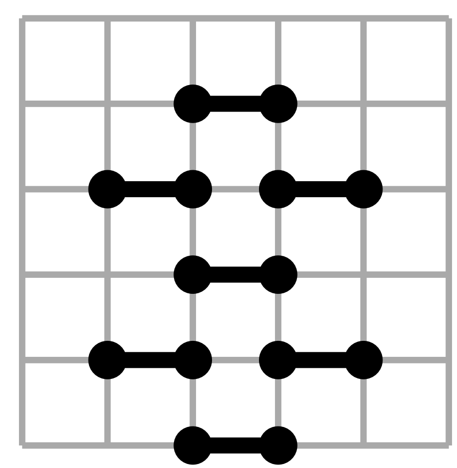
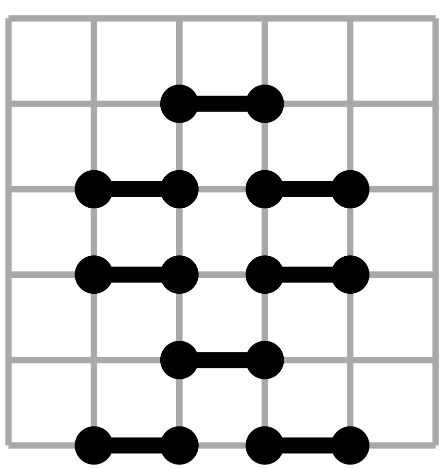
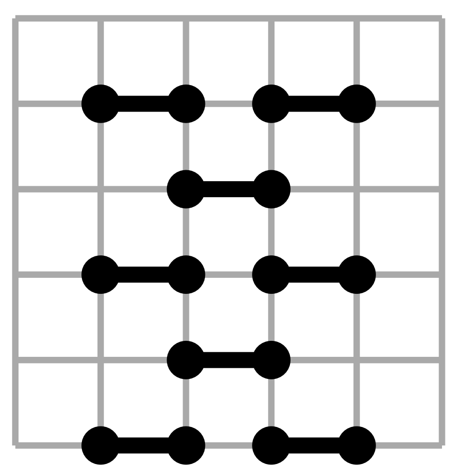
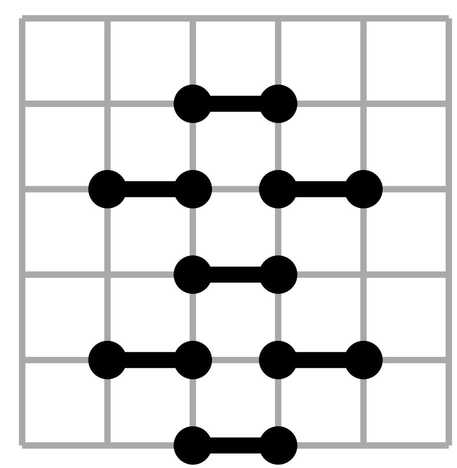
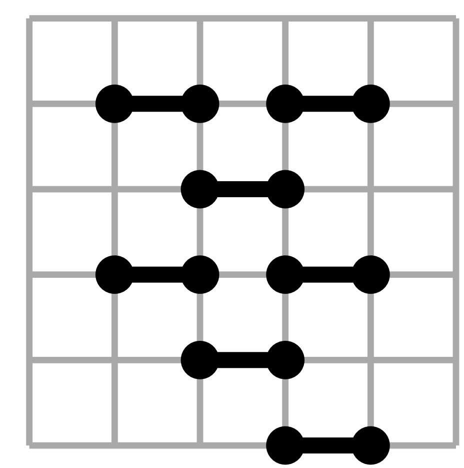
Connected heap	Strict heap	Pyramid
		
projection on the x-axis is connected	no two dimers on top of each other	only one minimal dimer

Heaps of dimers

Definition [Viennot, 1985]

Drop a finite number of dimers towards the x-axis until each dimer either touches the x-axis or another dimer.

Connected heap	**Strict heap**	**Pyramid**	**Half-pyramid**
			
projection on the x-axis is connected	no two dimers on top of each other	only one minimal dimer	no dimer right of minimal dimer

Viennot's bijection

[Viennot, 1985]

Directed animals on the square grid correspond to strict pyramids:

Viennot's bijection

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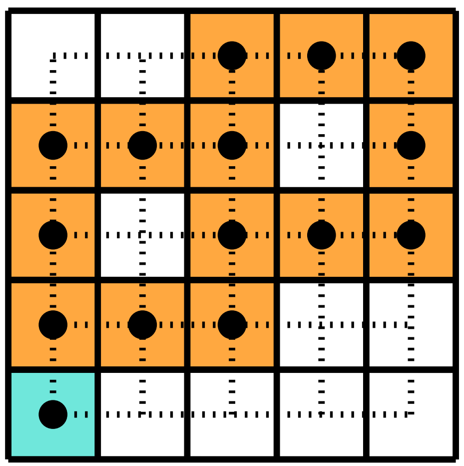

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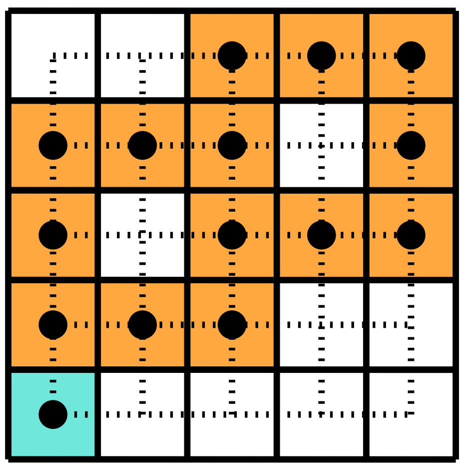
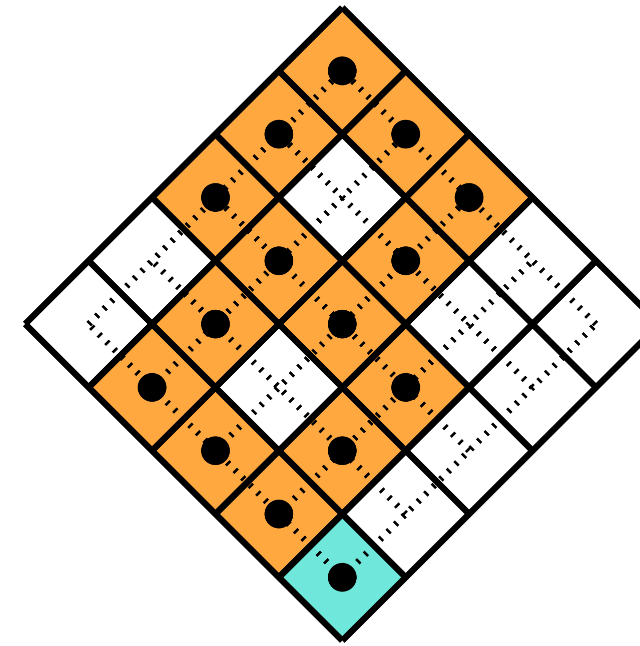


Rotate by 45 degrees
counter-clockwise



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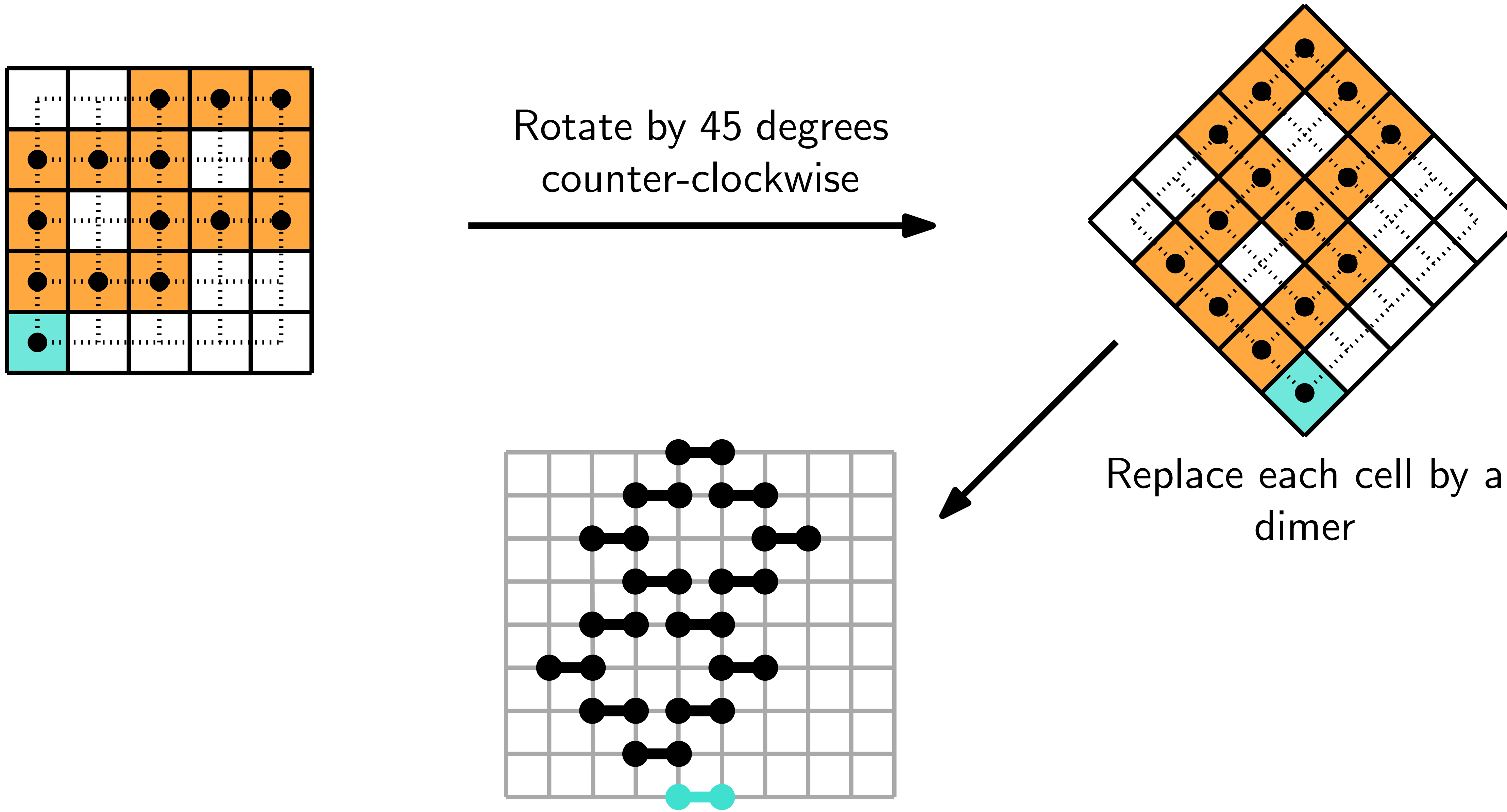
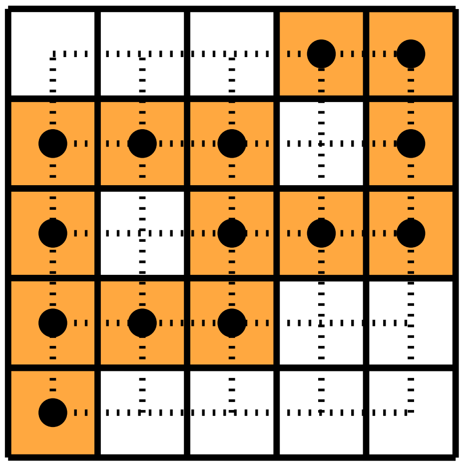


Extension to general lattice animals

[Bousquet-Mélou & Rechnitzer, 2002]



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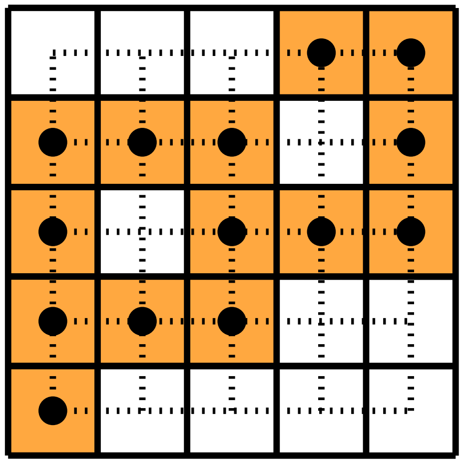

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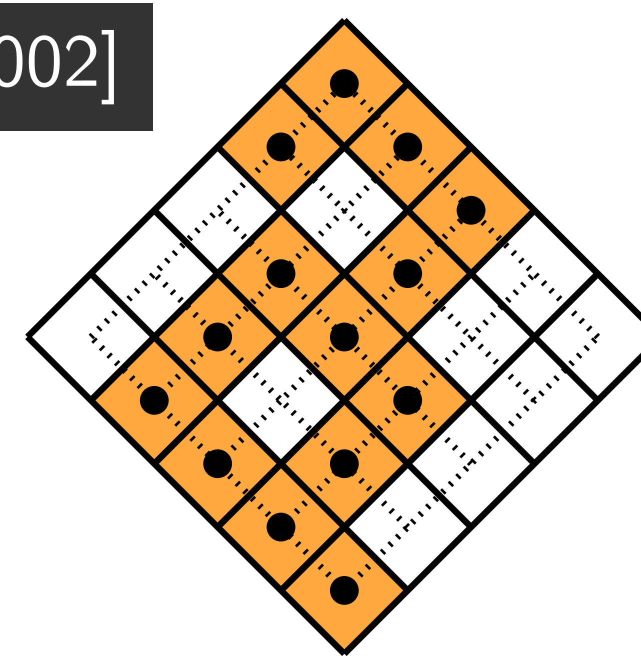

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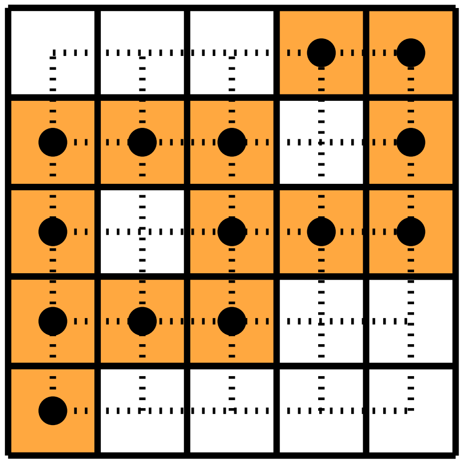

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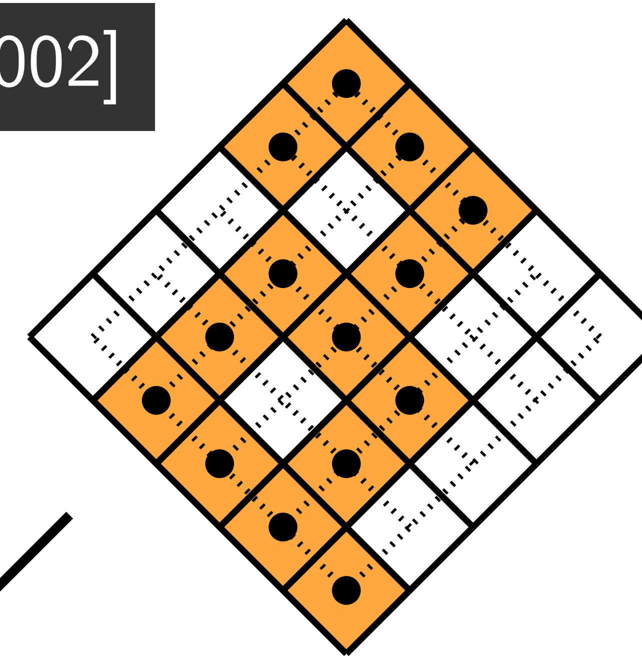
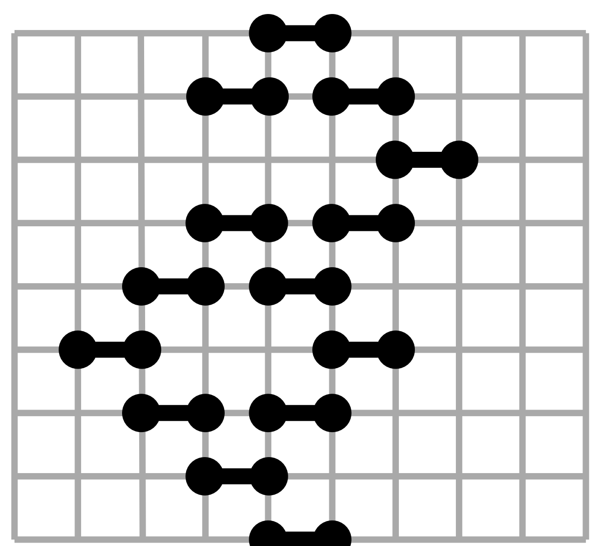


Replace each cell by a dimer



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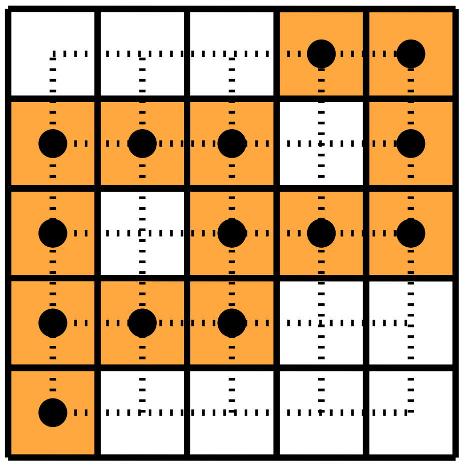

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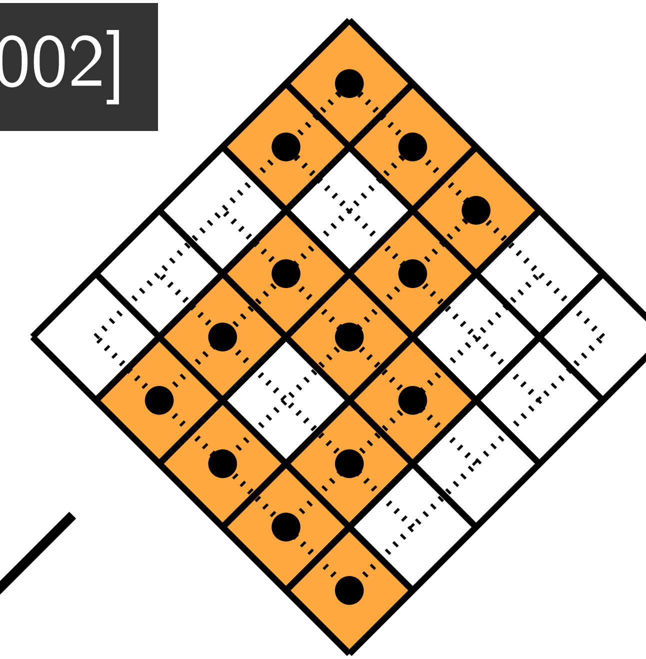
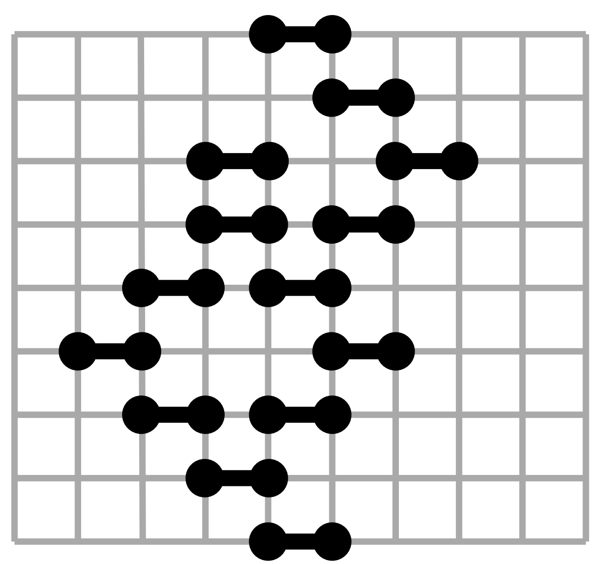


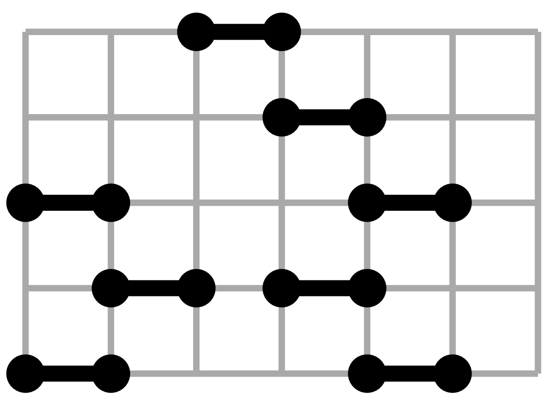
Replace each cell by a
dimer

Let the dimers drop!



Reversing the procedure

- Every square lattice animal maps to a connected heap
- Not every connected heap is the image of a square lattice animal!
- But every strict connected heap is the image of a square lattice animal!



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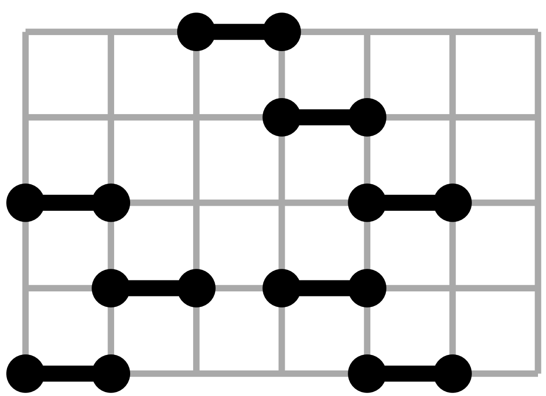
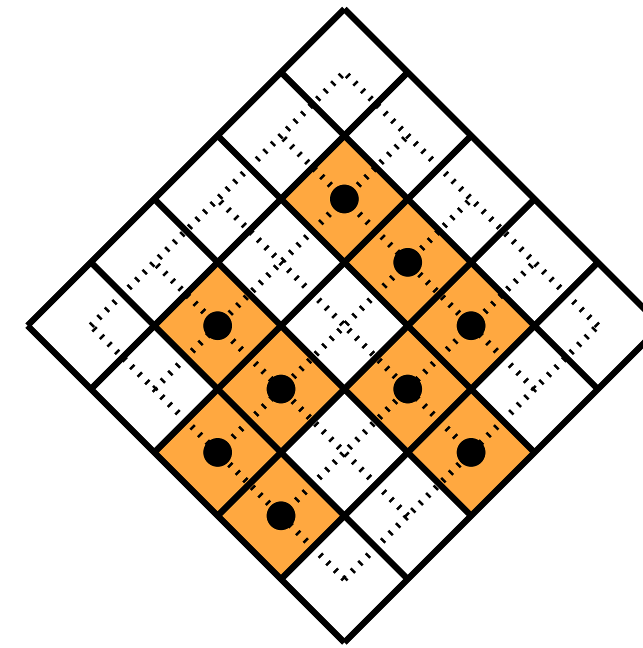


Start with a strict, connected heap H

Replace each cell by a dimer



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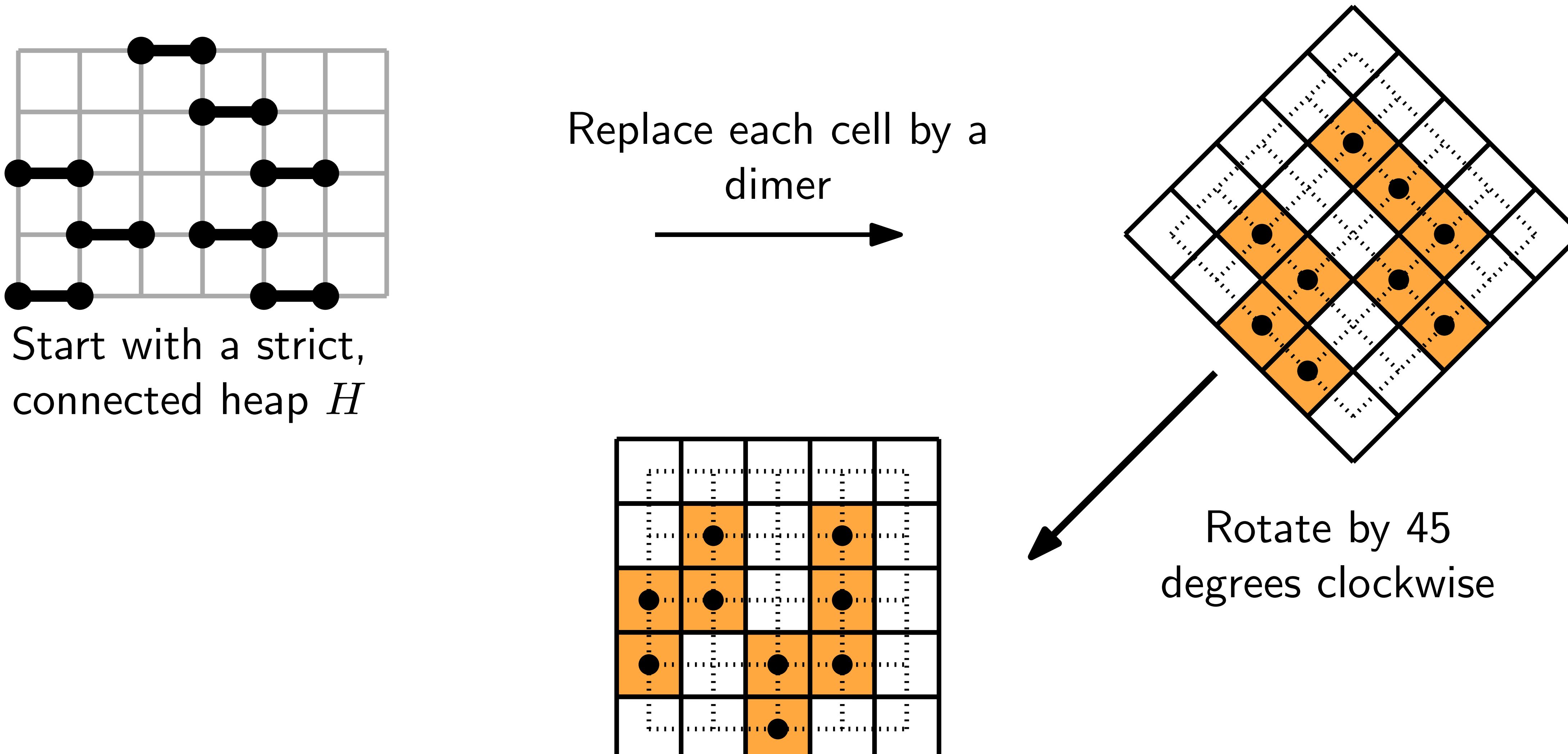

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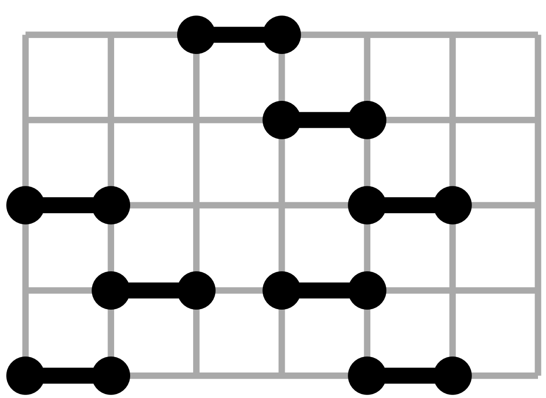
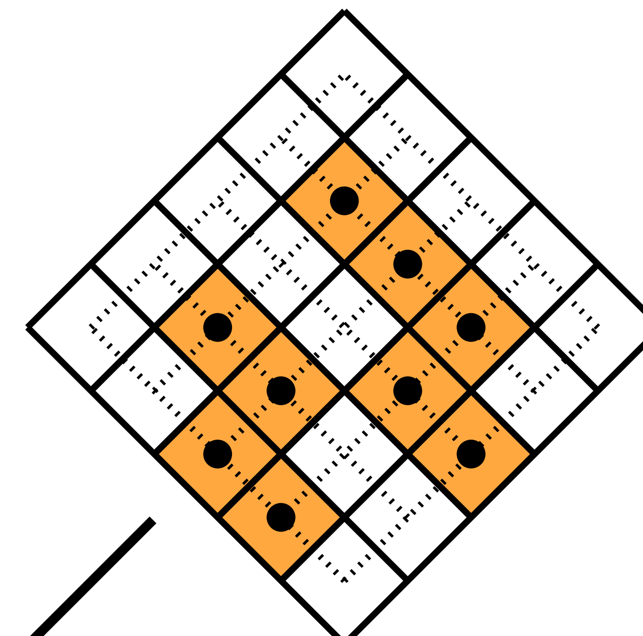
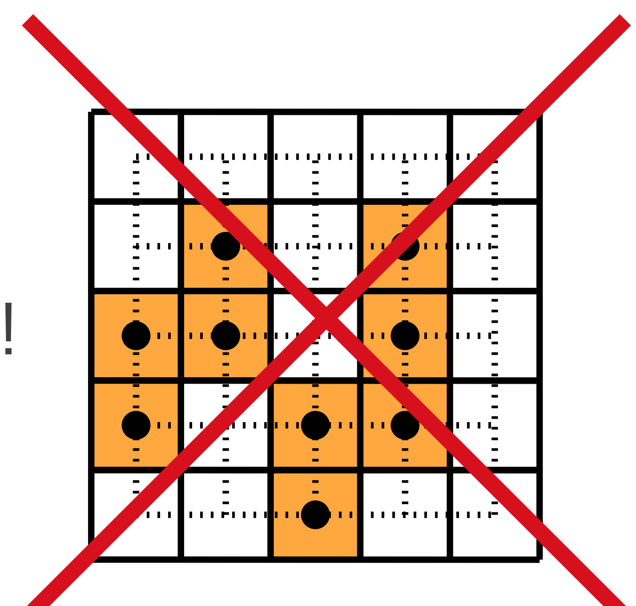


Start with a strict, connected heap H

Replace each cell by a dimer



Rotate by 45 degrees clockwise



Not a lattice animal anymore!

Reversing the procedure

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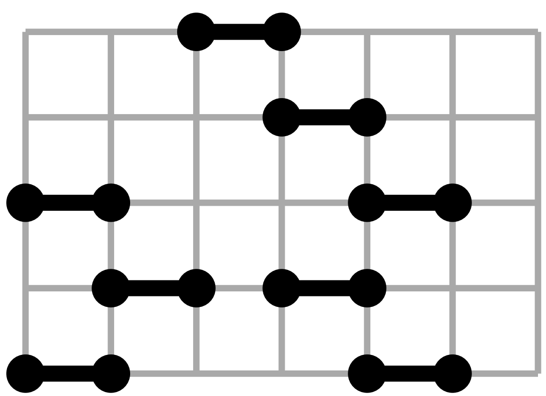

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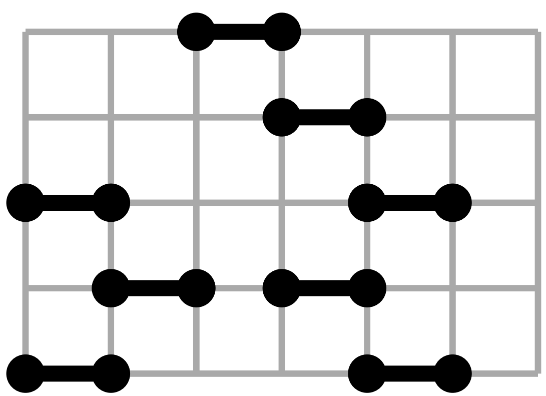
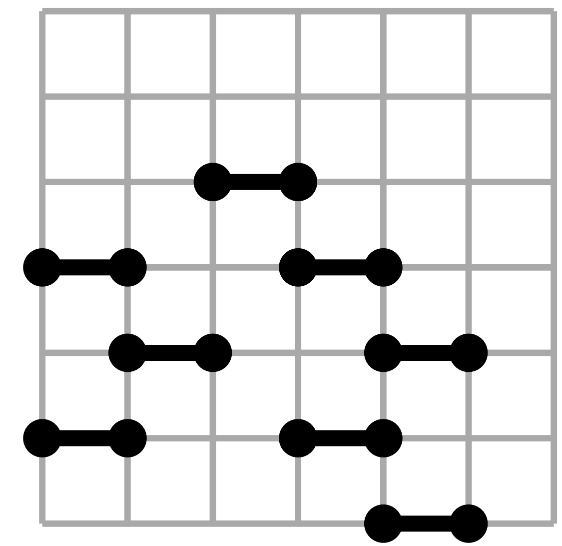


Start with a strict, connected heap H

Push the left-most minimal dimer up $\longrightarrow$



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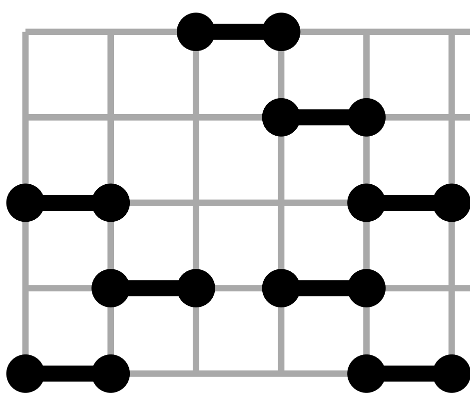

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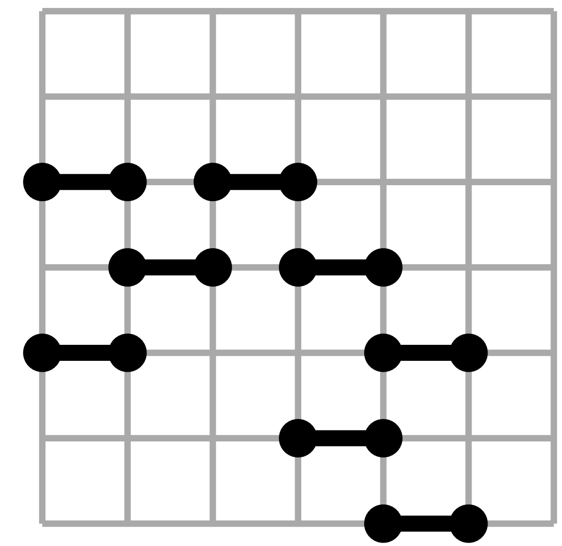

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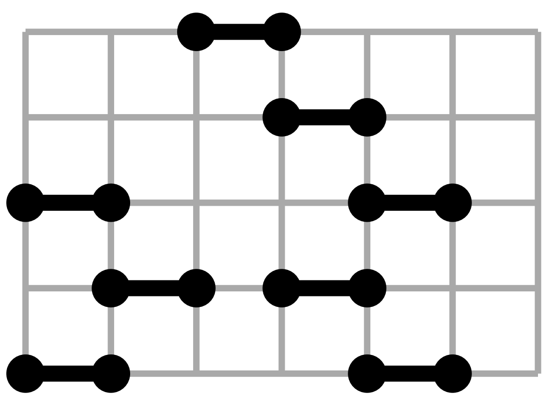

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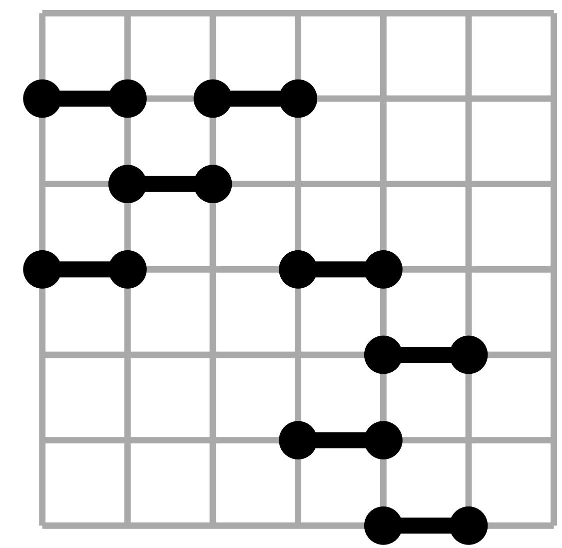

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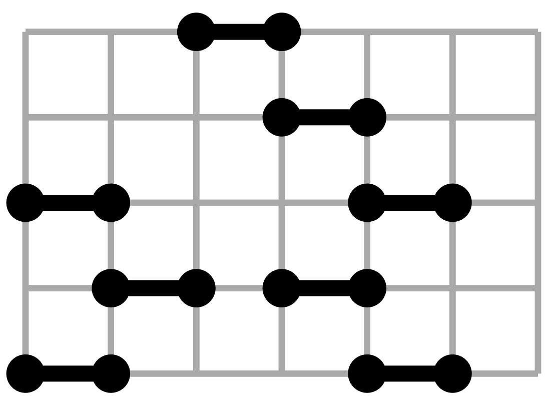

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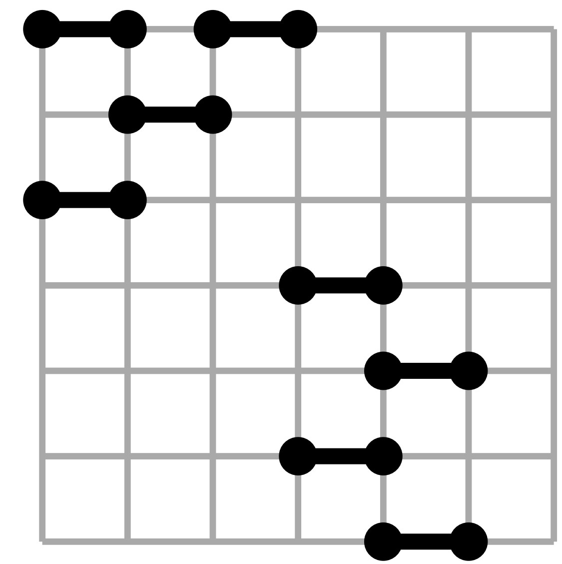

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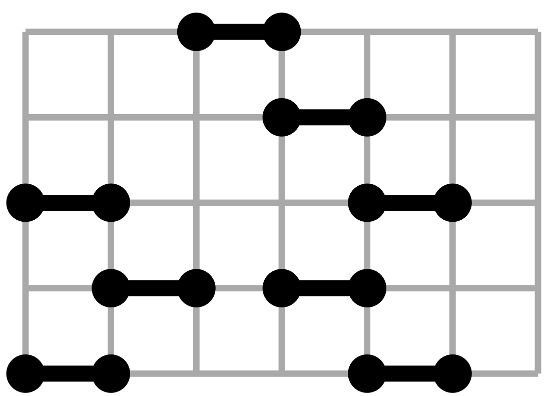

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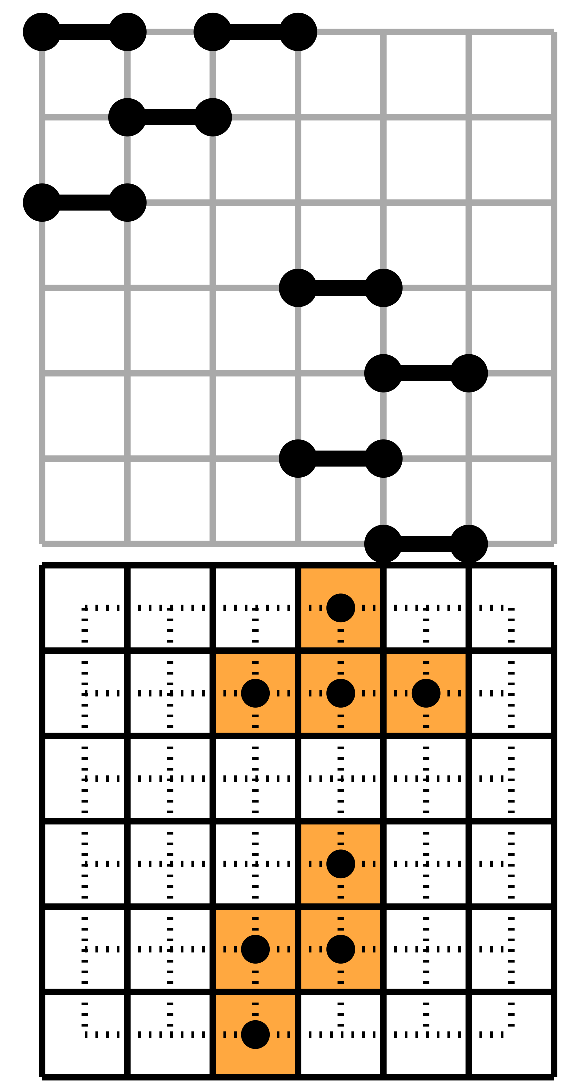

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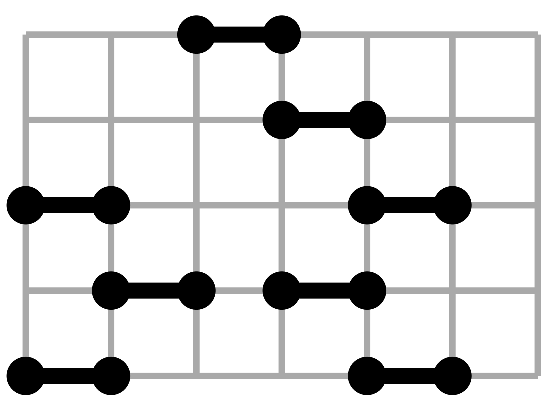

Start with a strict, connected heap H

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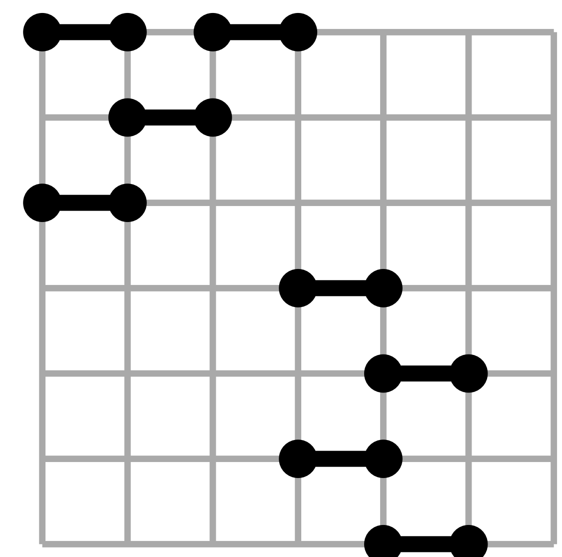
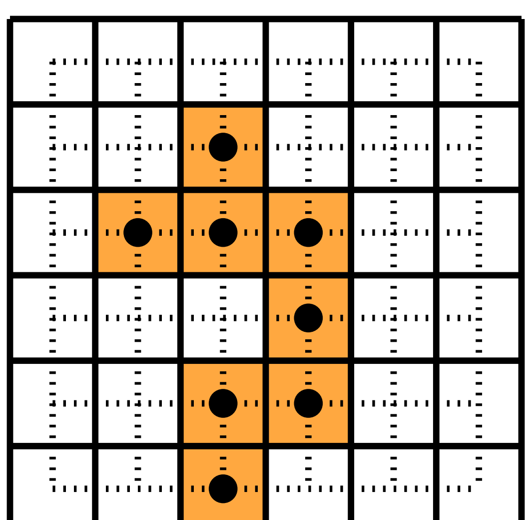
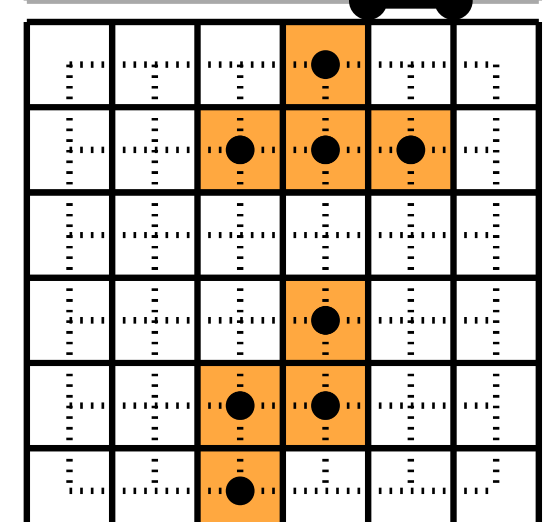
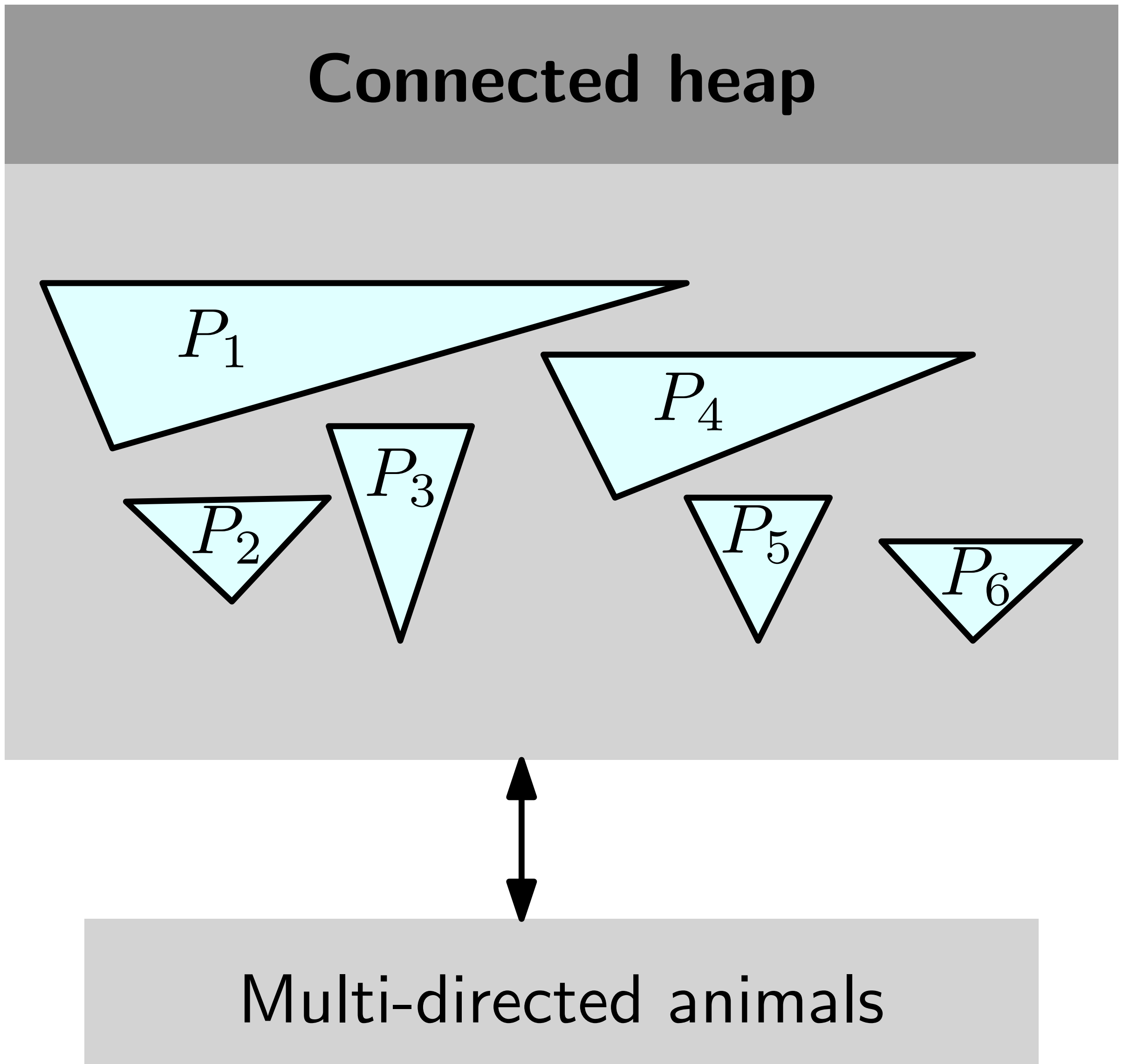




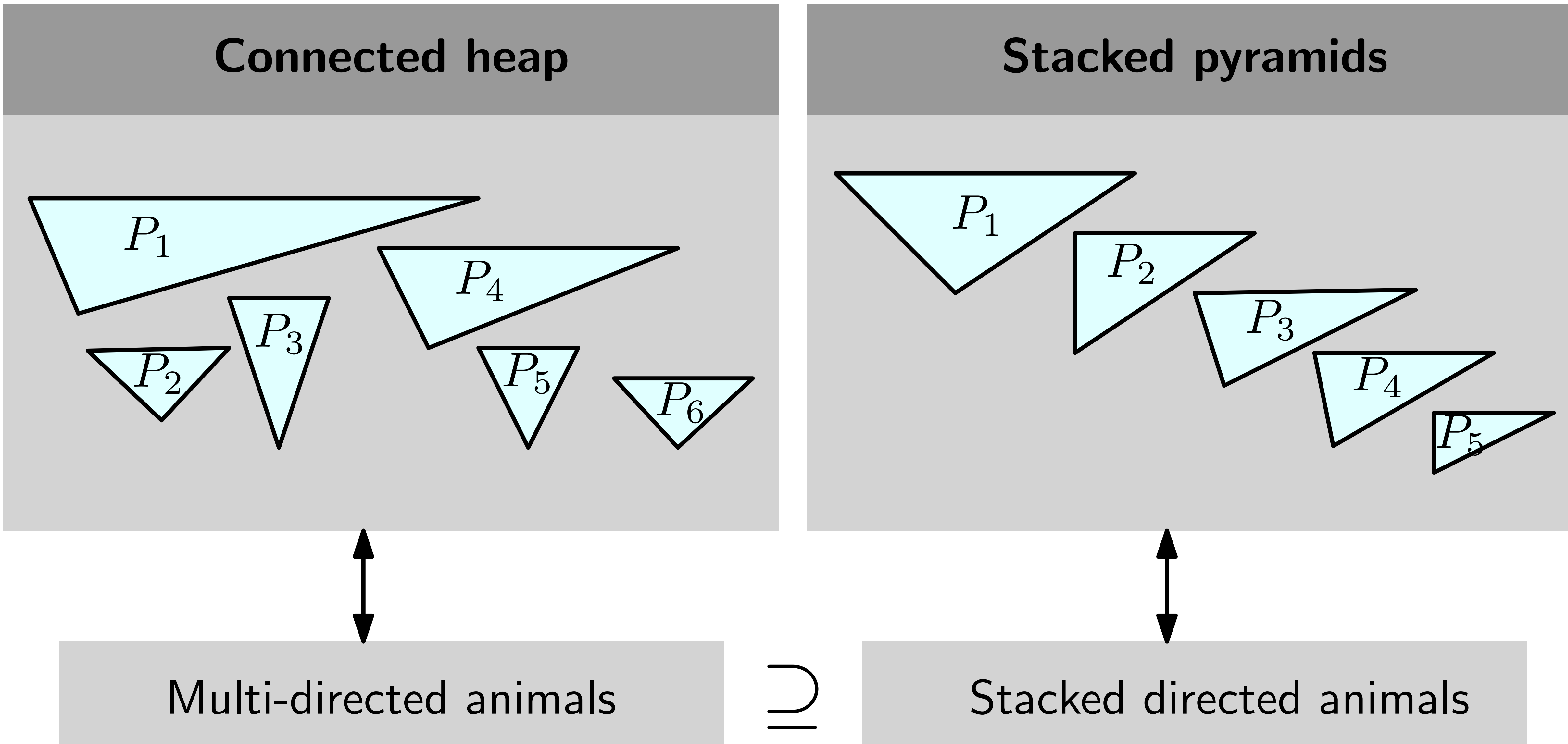
Push the upper directed animal down until it connects to the lower directed animal



Stacked directed animals

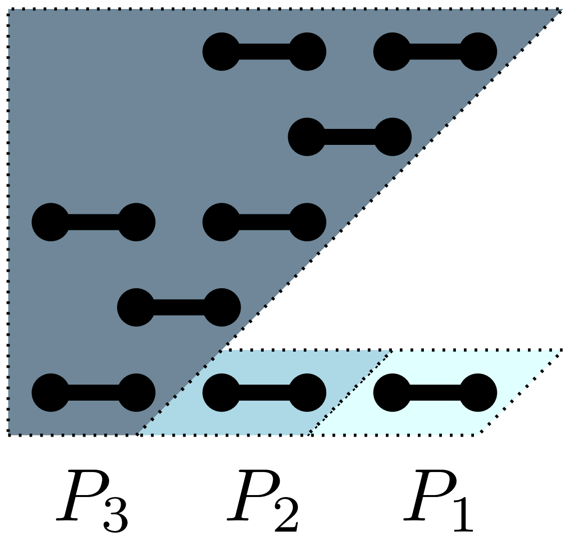


Stacked directed animals

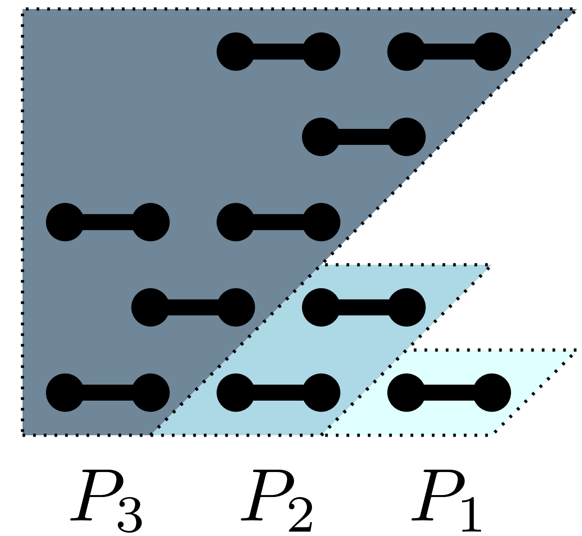


Stacked directed animals. Example

Connected heap

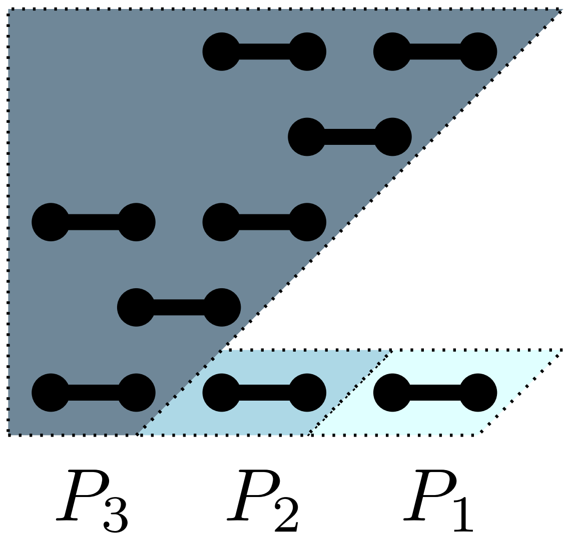


Stacked pyramids



Stacked directed animals. Example

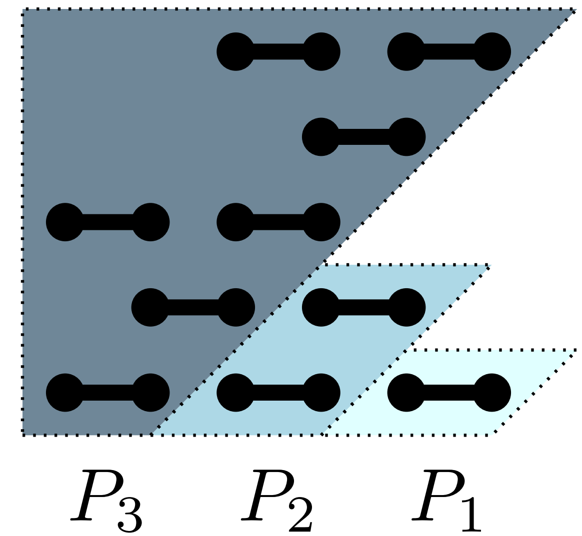
Connected heap



$\pi(P_3) \cap \pi(P_2) \neq \emptyset$ ✓

$\pi(P_3) \cap \pi(P_2) \neq \emptyset$ ✗

Stacked pyramids

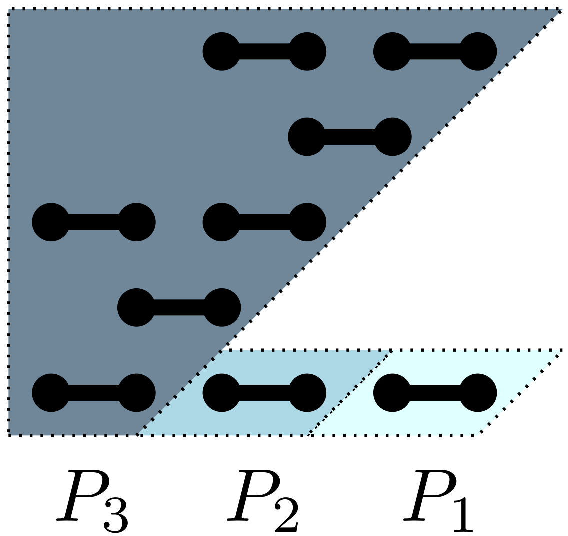


$\pi(P_3) \cap \pi(P_2) \neq \emptyset$ ✓

$\pi(P_2) \cap \pi(P_1) \neq \emptyset$ ✓

Stacked directed animals. Example

Connected heap



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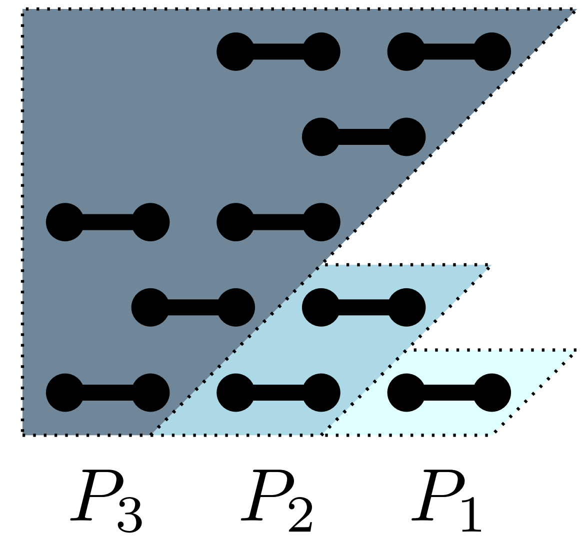
Stacked pyramids



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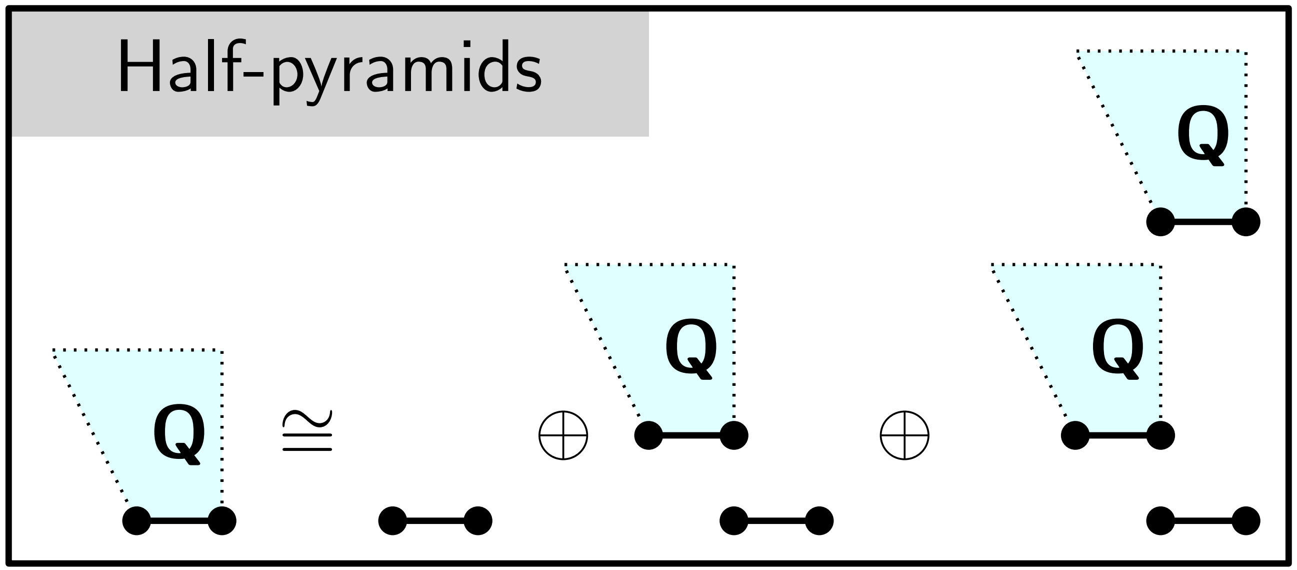
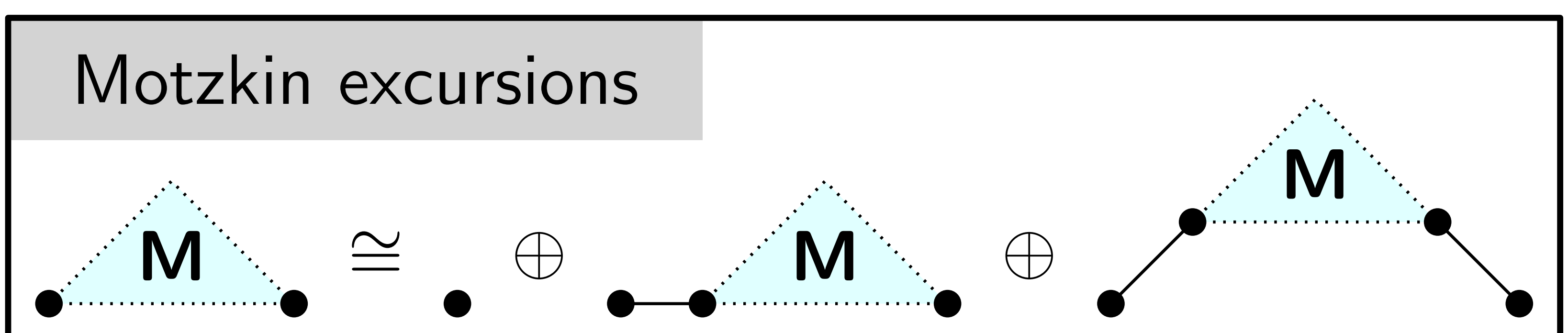
$\pi(P_2) \cap \pi(P_1) \neq \emptyset$

Half-pyramids ≅ Motzkin excursions

Note the structurally equivalent symbolic descriptions:

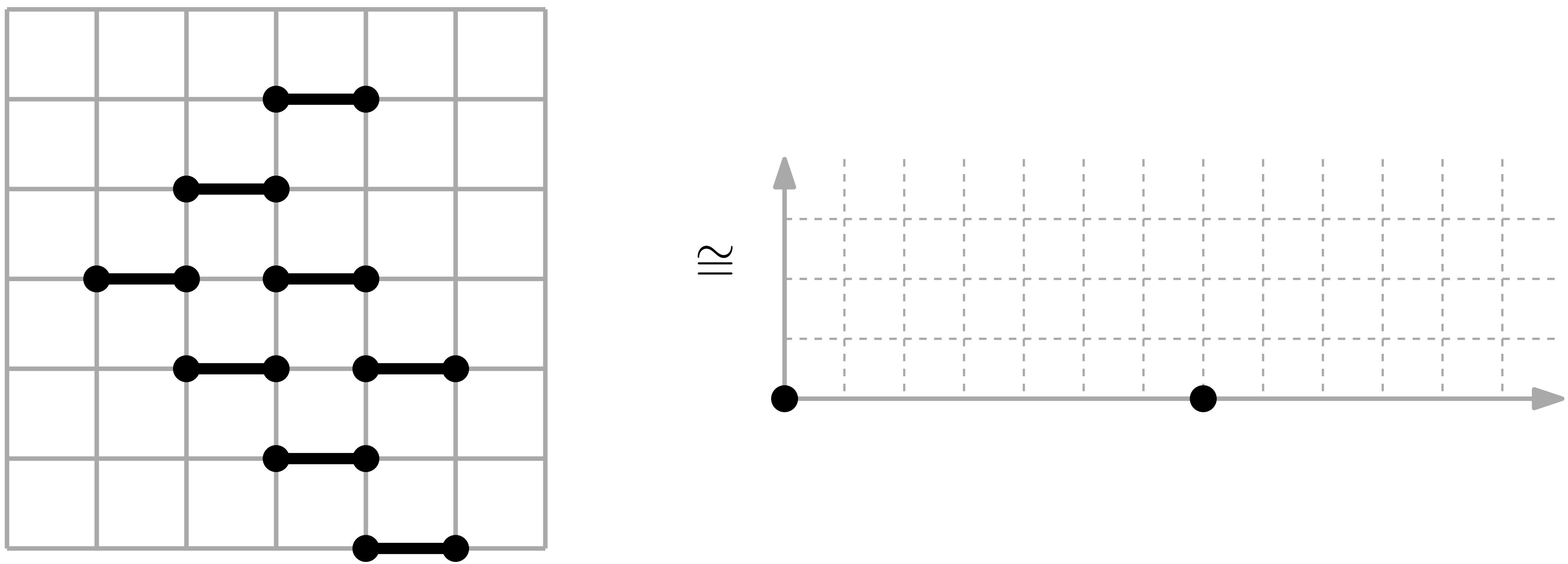


[Bousquet-Mélou & Rechnitzer, 2002]



Half-pyramids $\cong$ Motzkin excursions

Consider an example:



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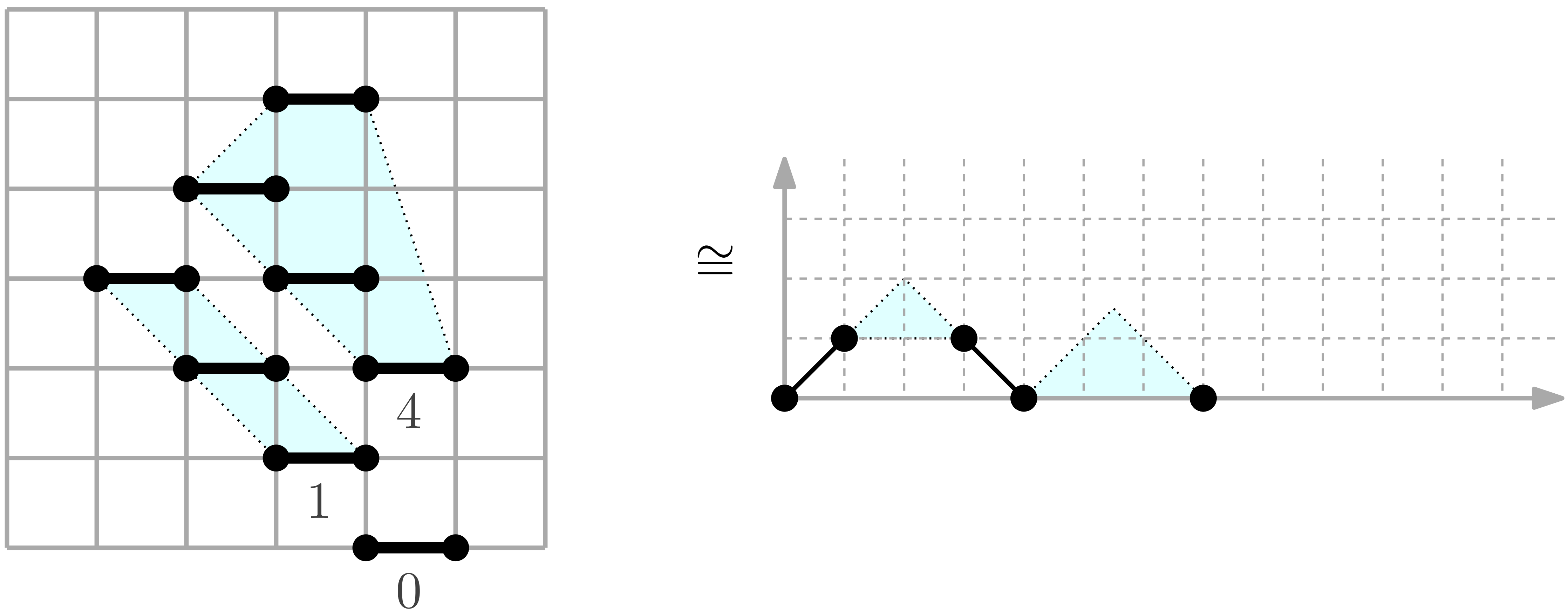

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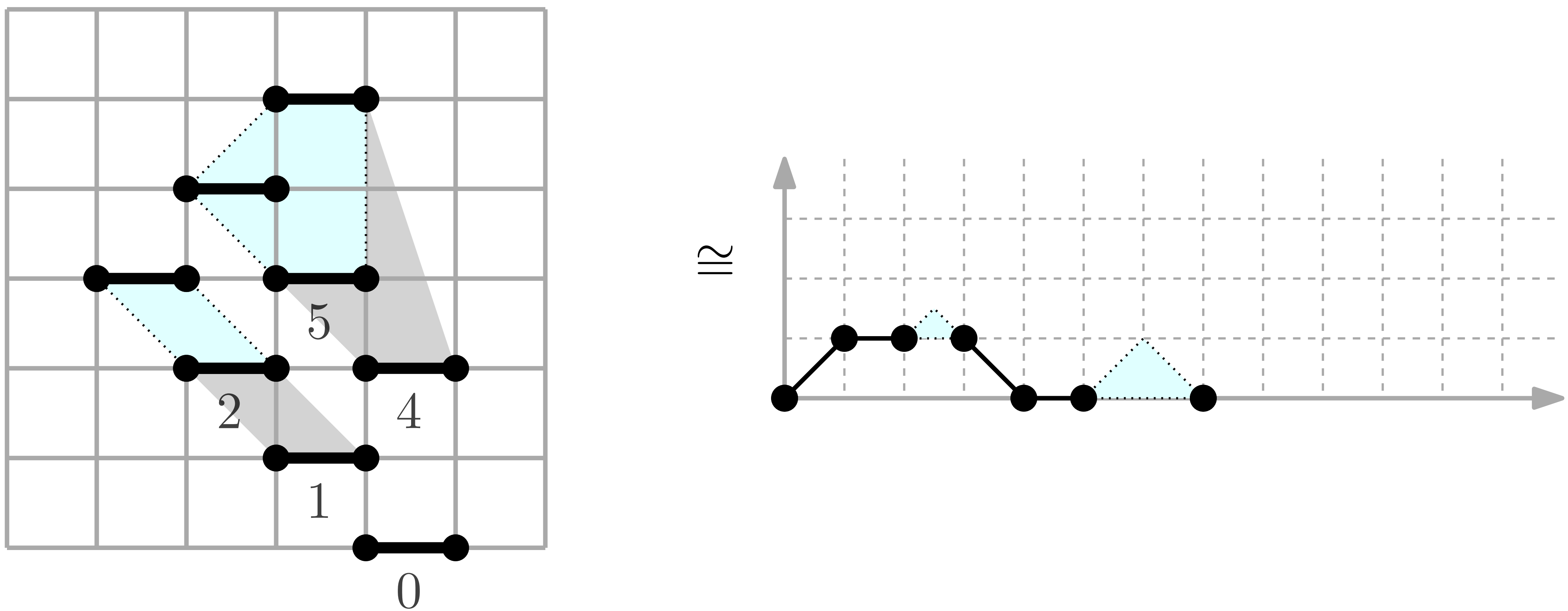

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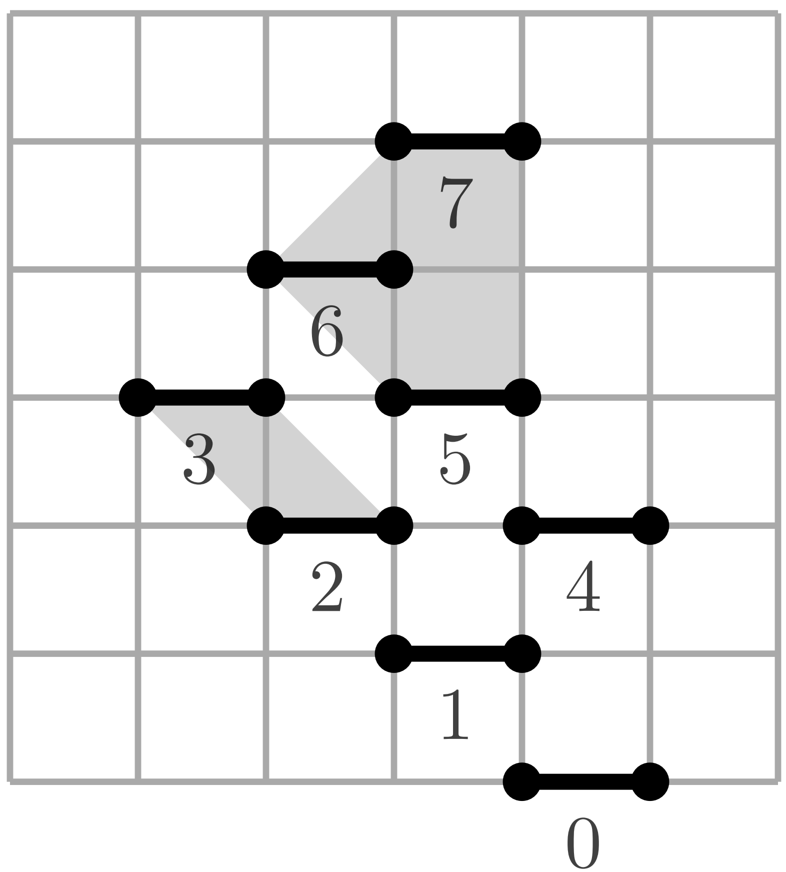
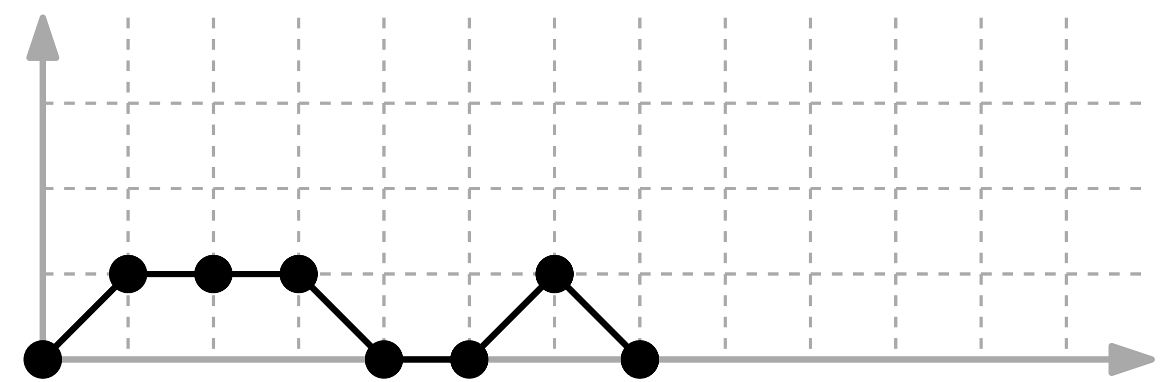
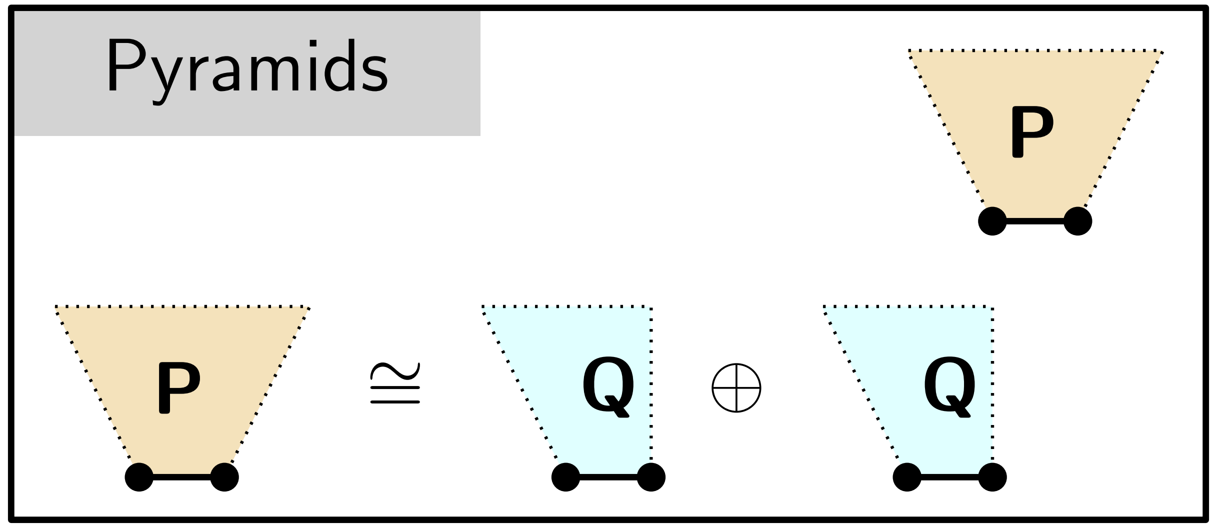


$\cong$



Pyramids $\cong$ Motzkin excursions with catastrophes only at the x-axis



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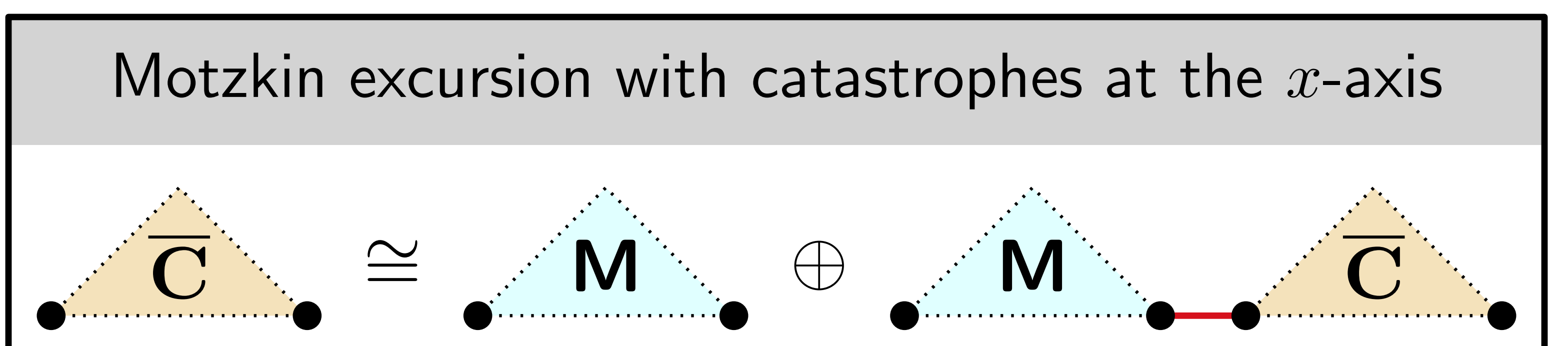

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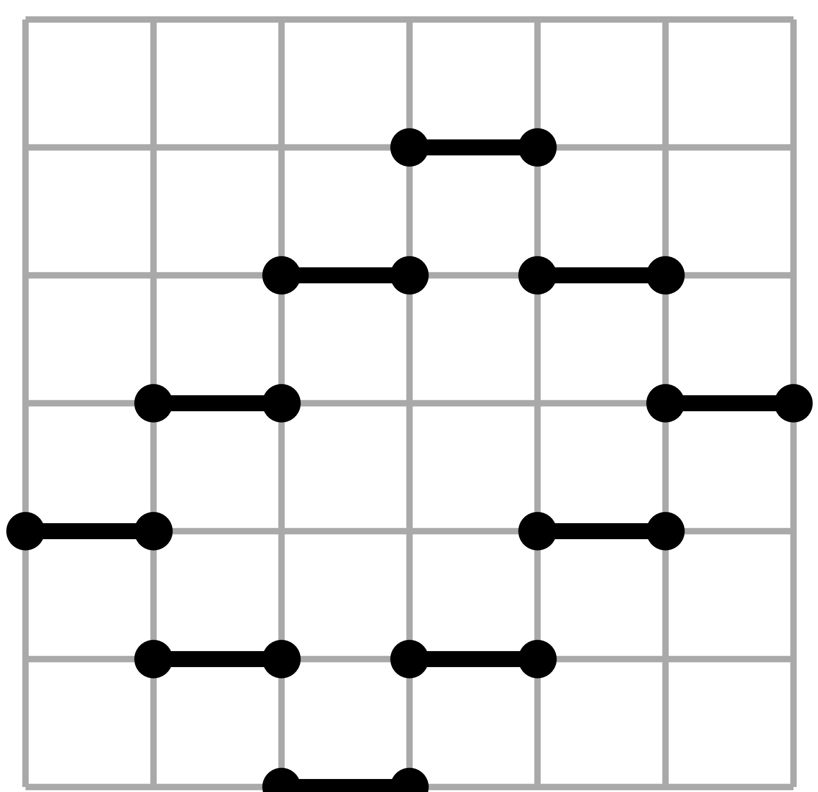
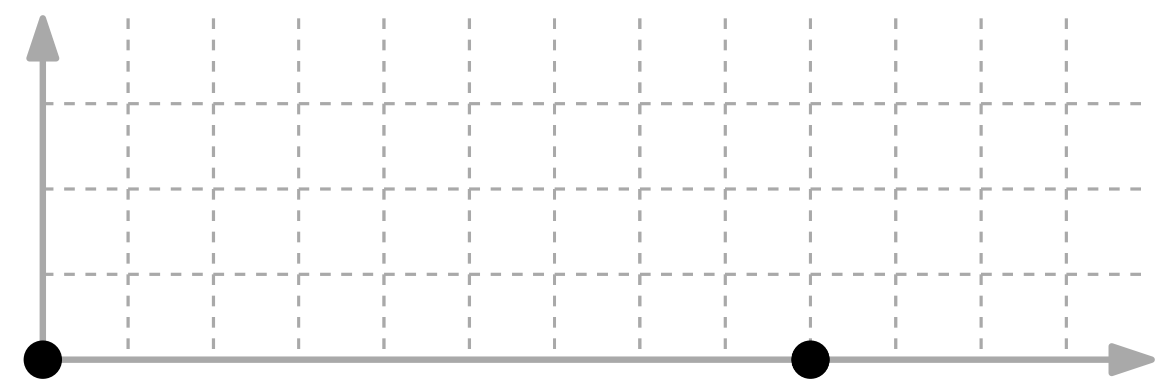


$\cong$



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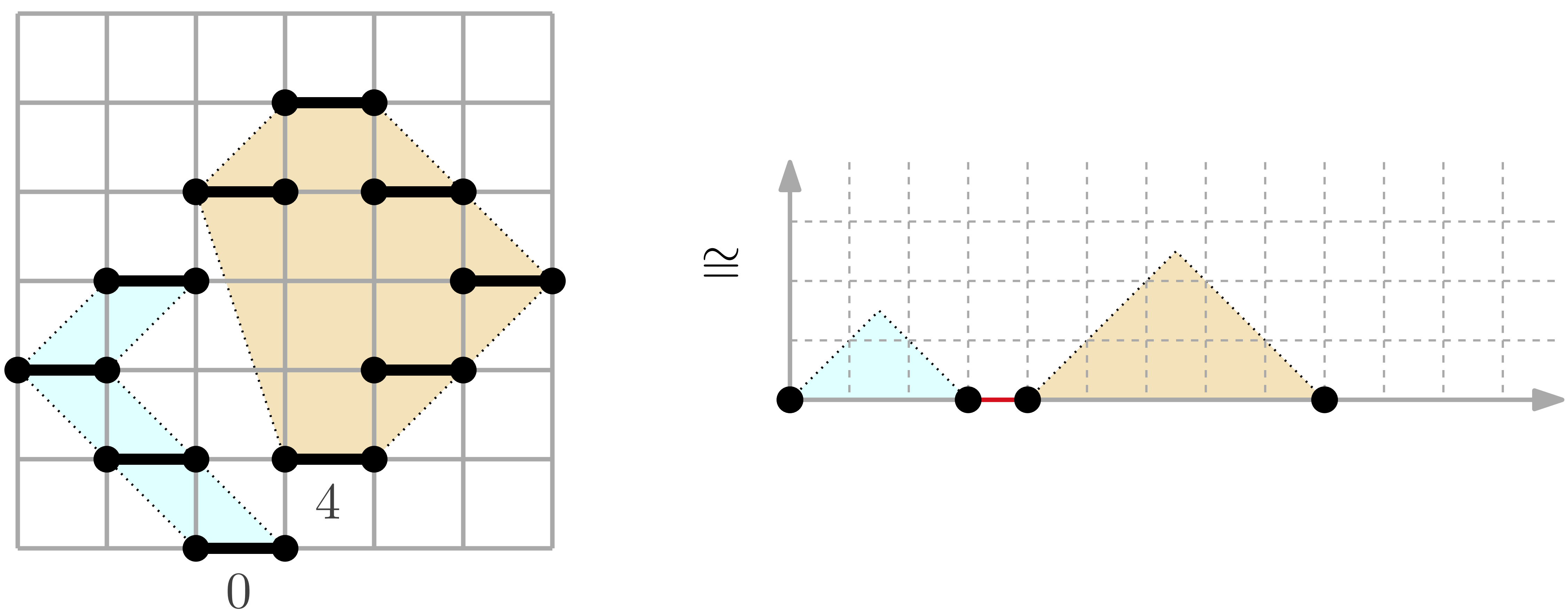

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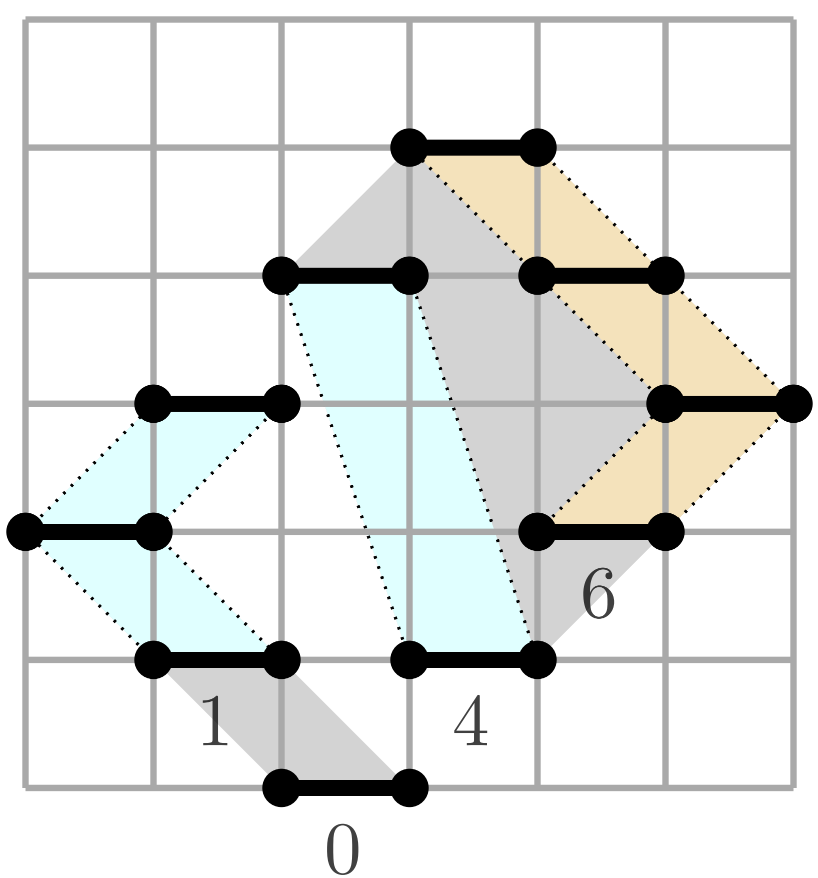

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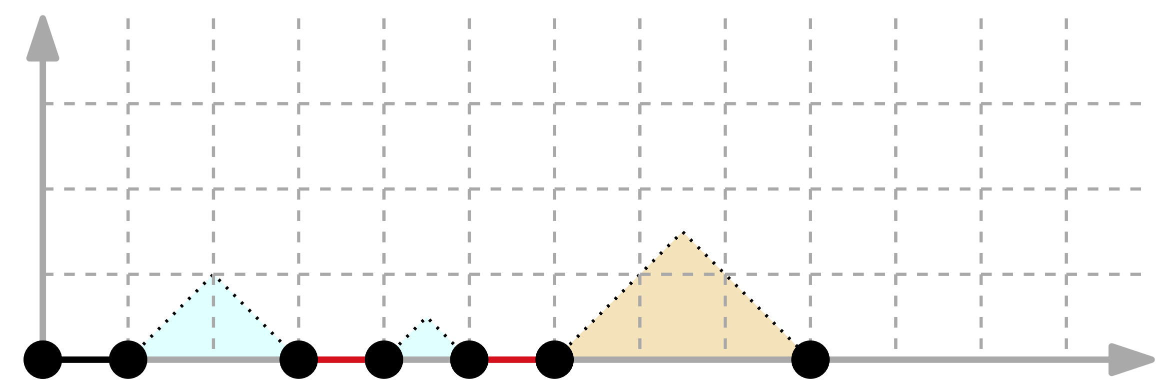

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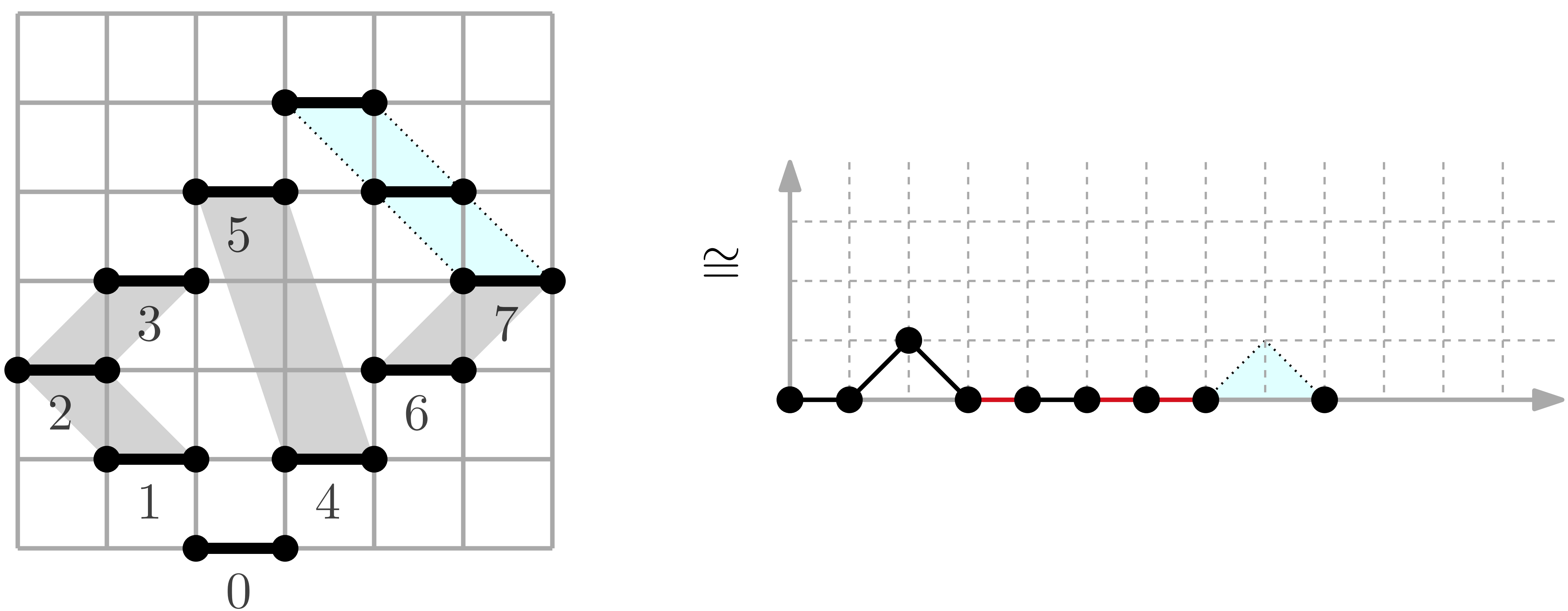

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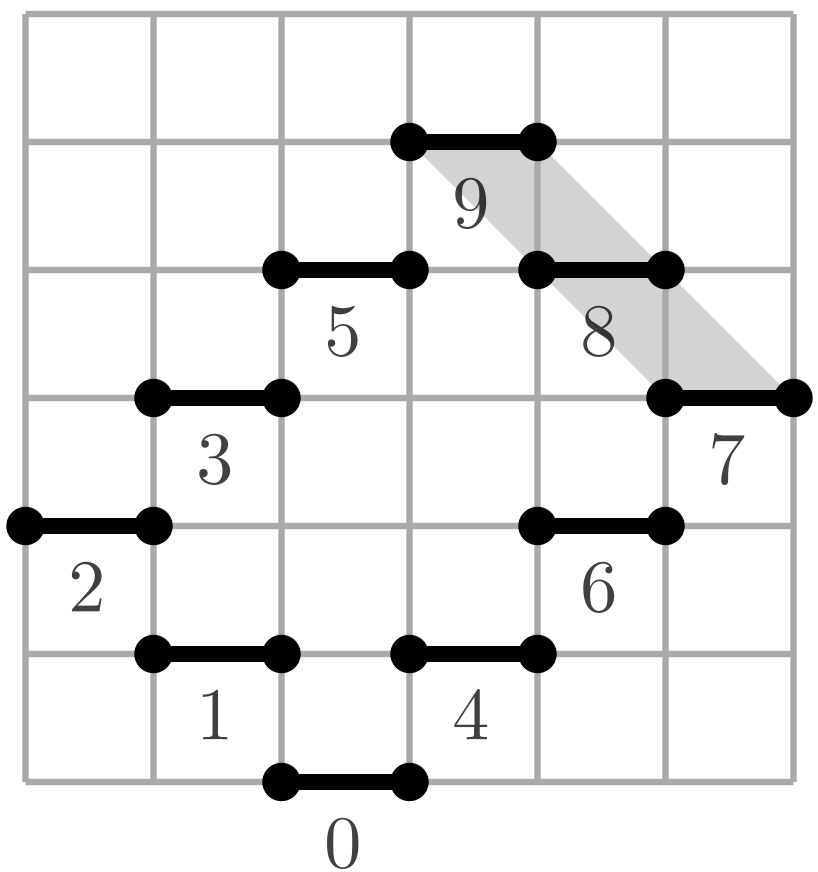

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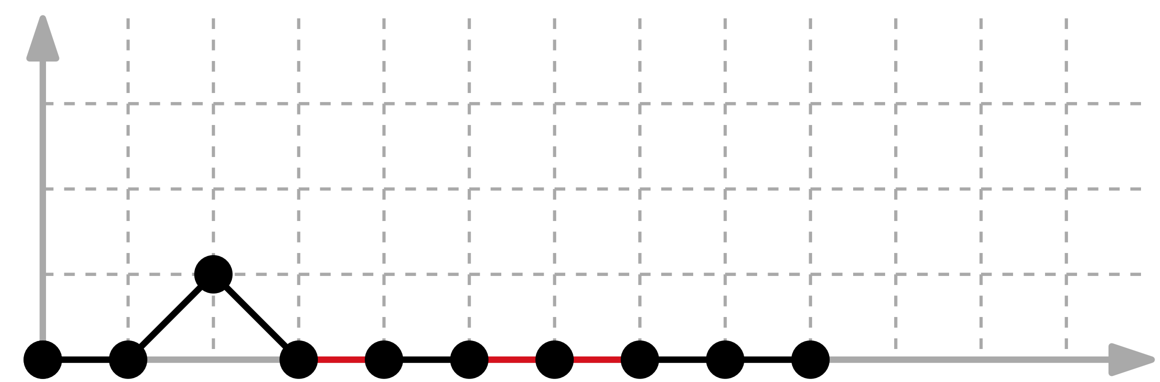
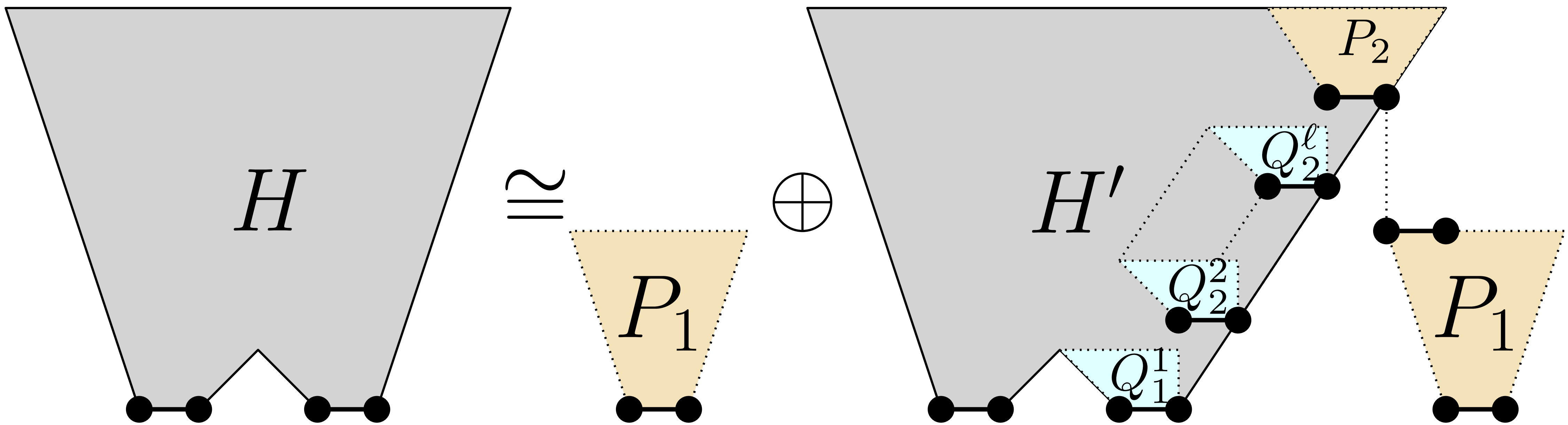
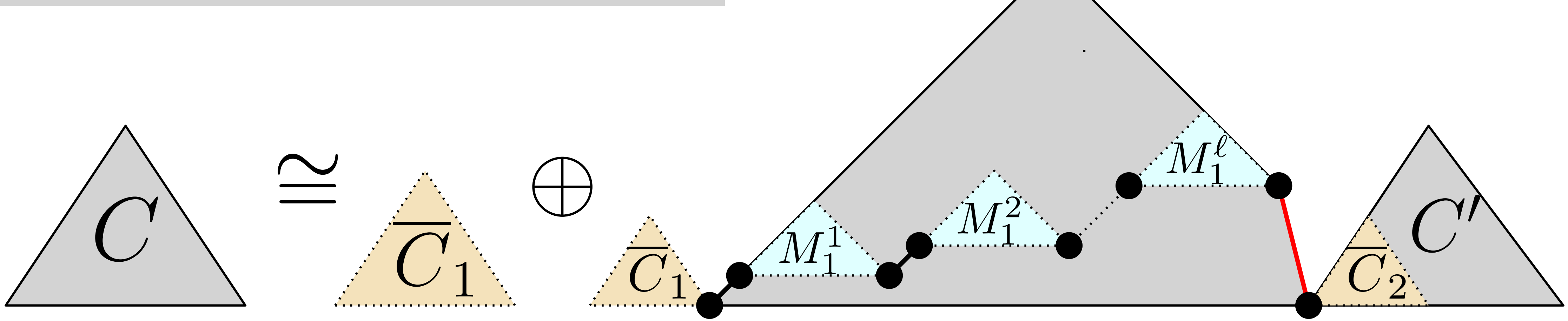


Stacked directed animals ≅ Motzkin excursions with catastrophes

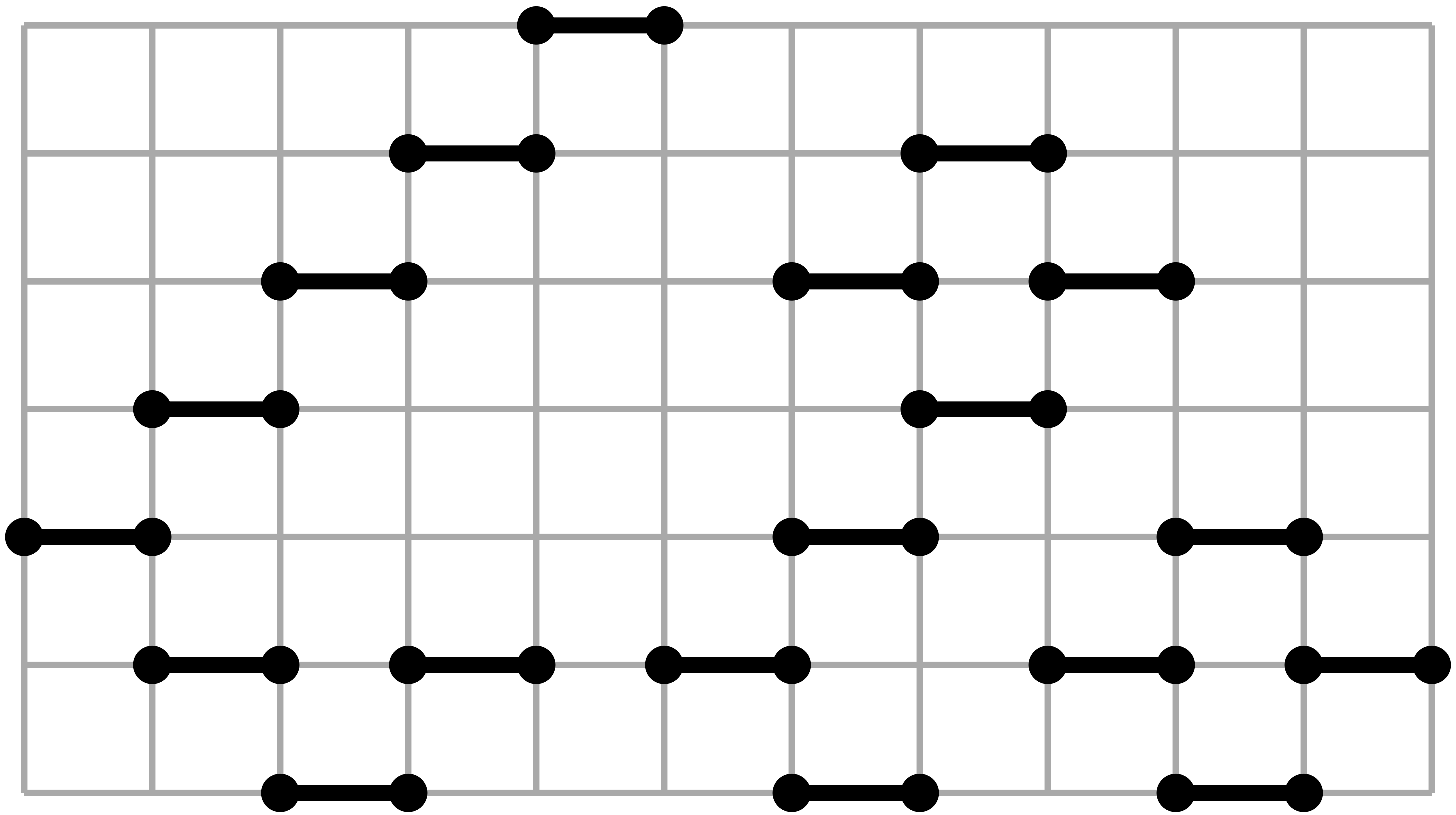
Stacked pyramids

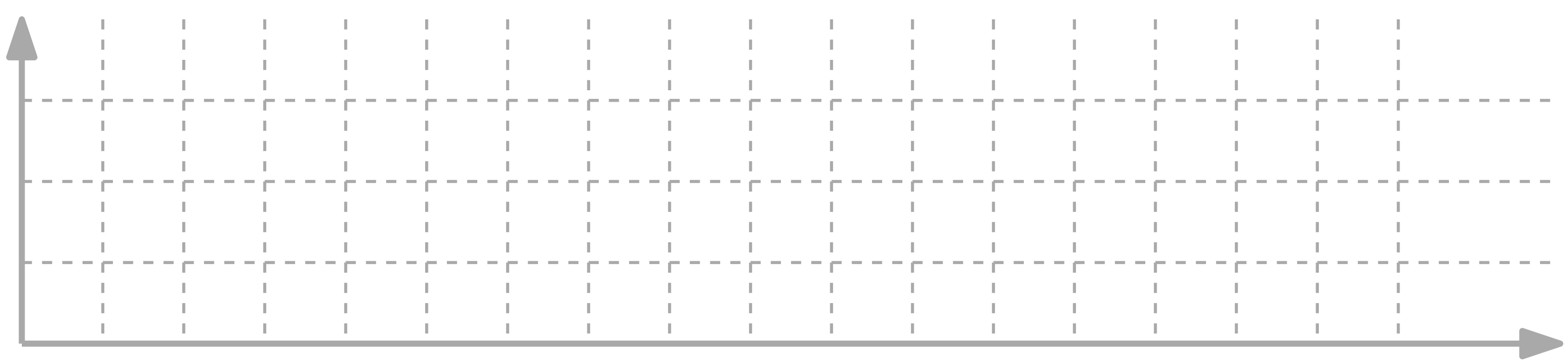


Motzkin excursions with catastrophes



Bijection. Example





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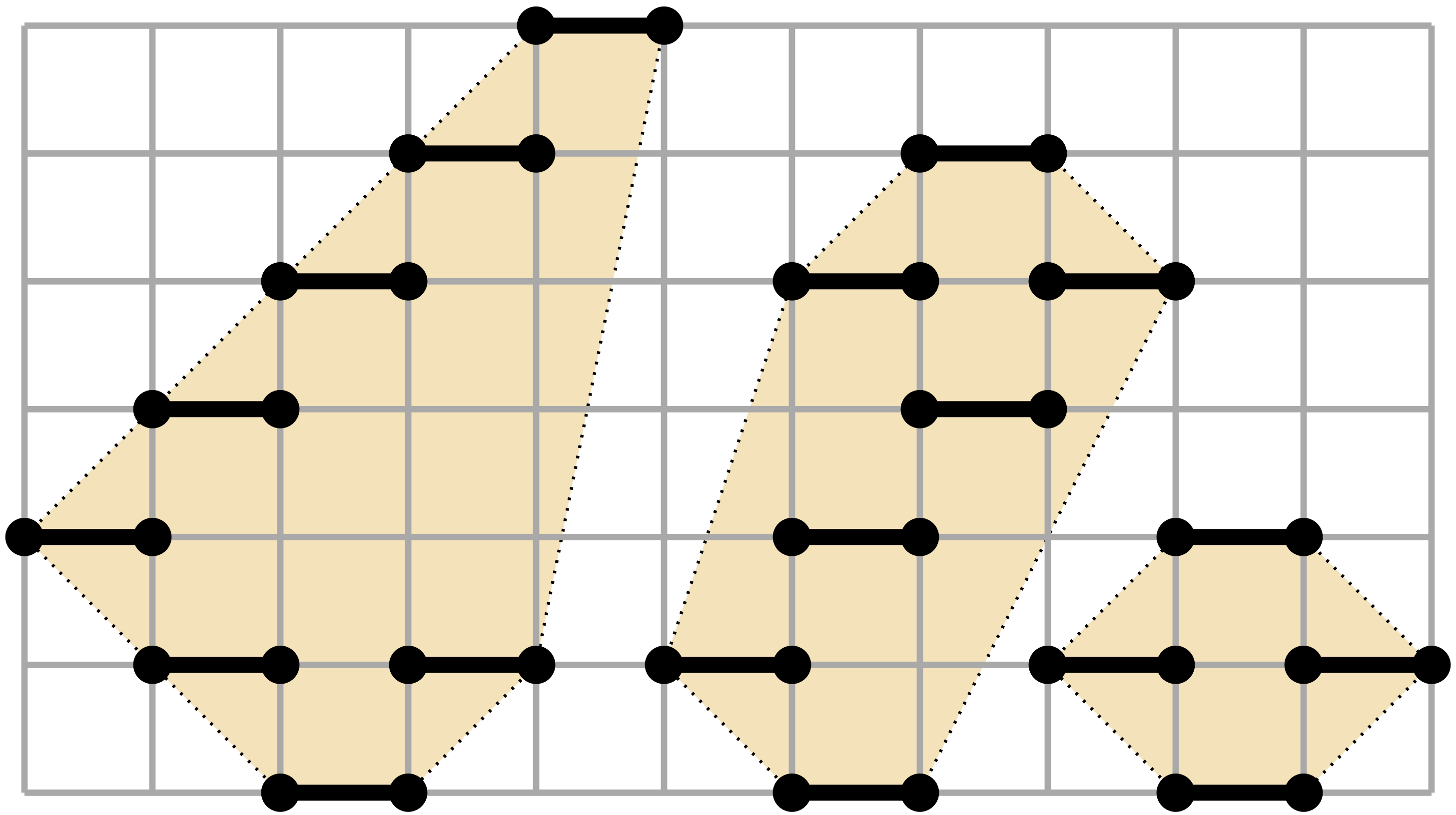

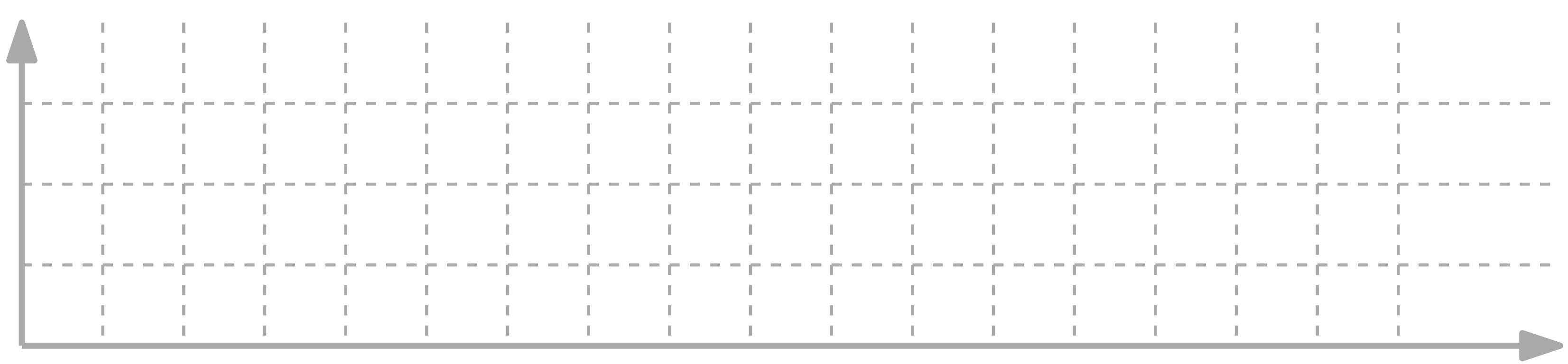

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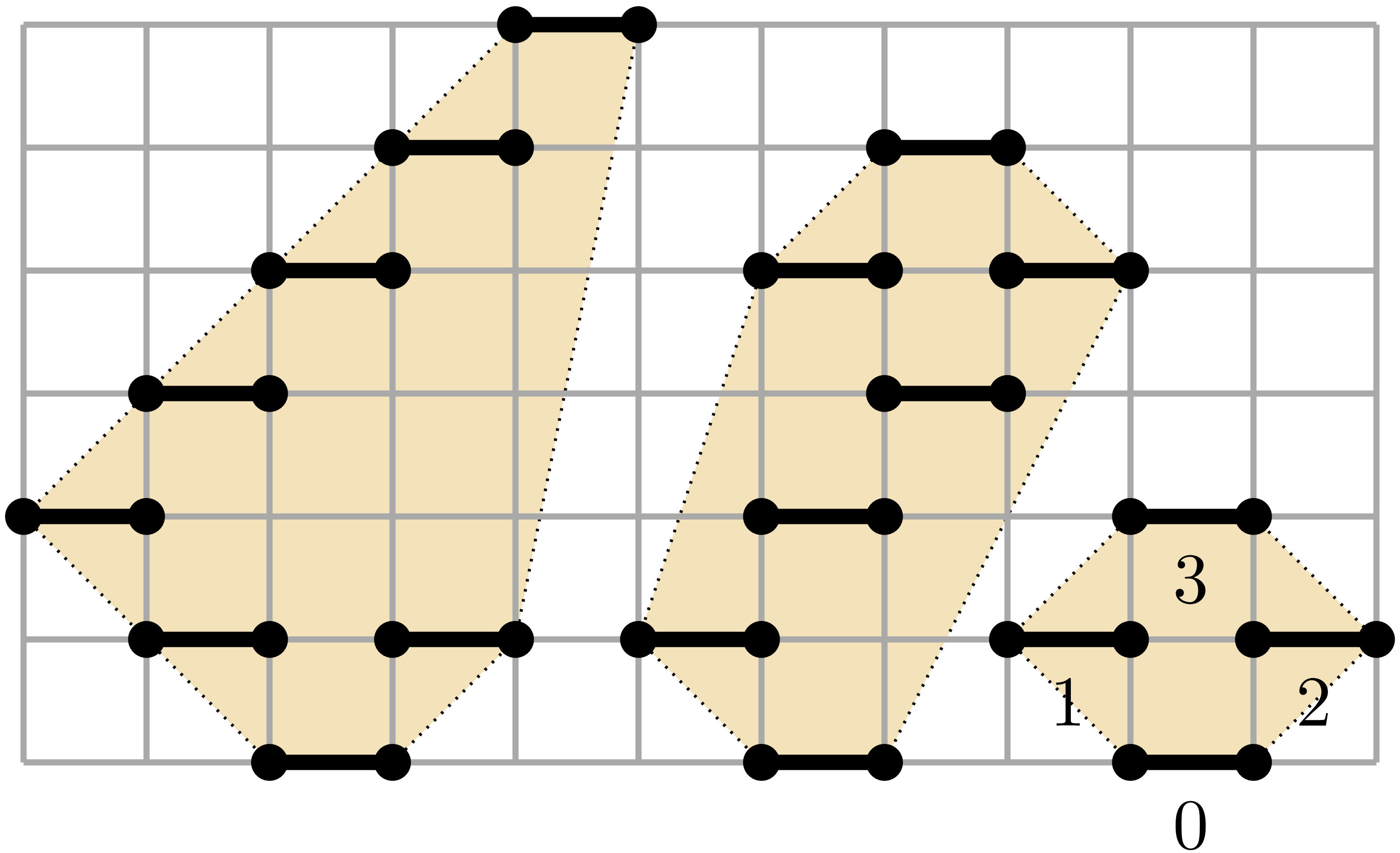

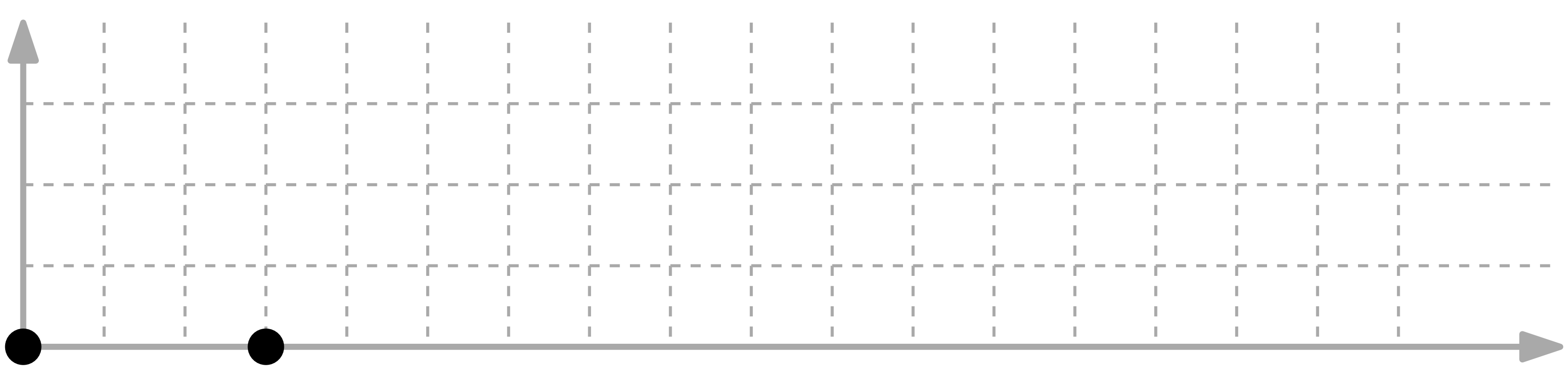

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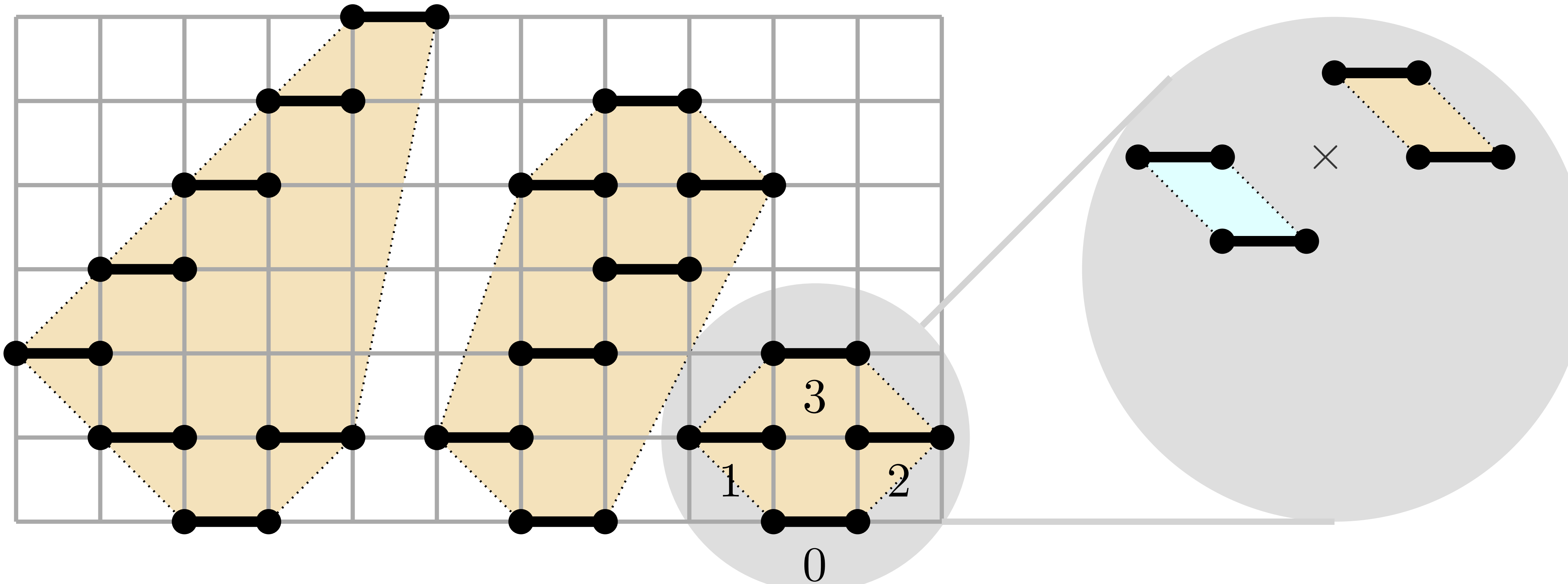

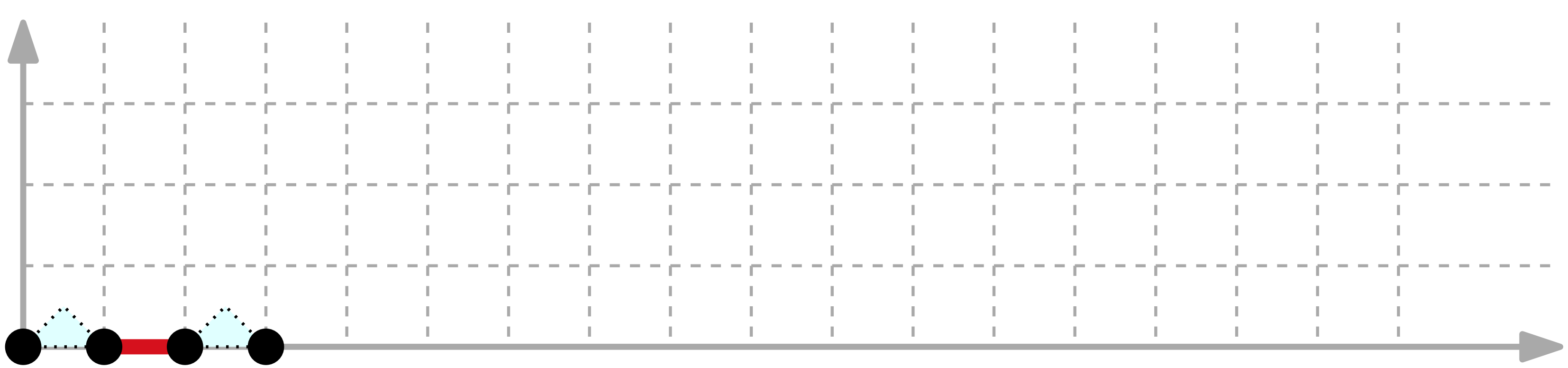

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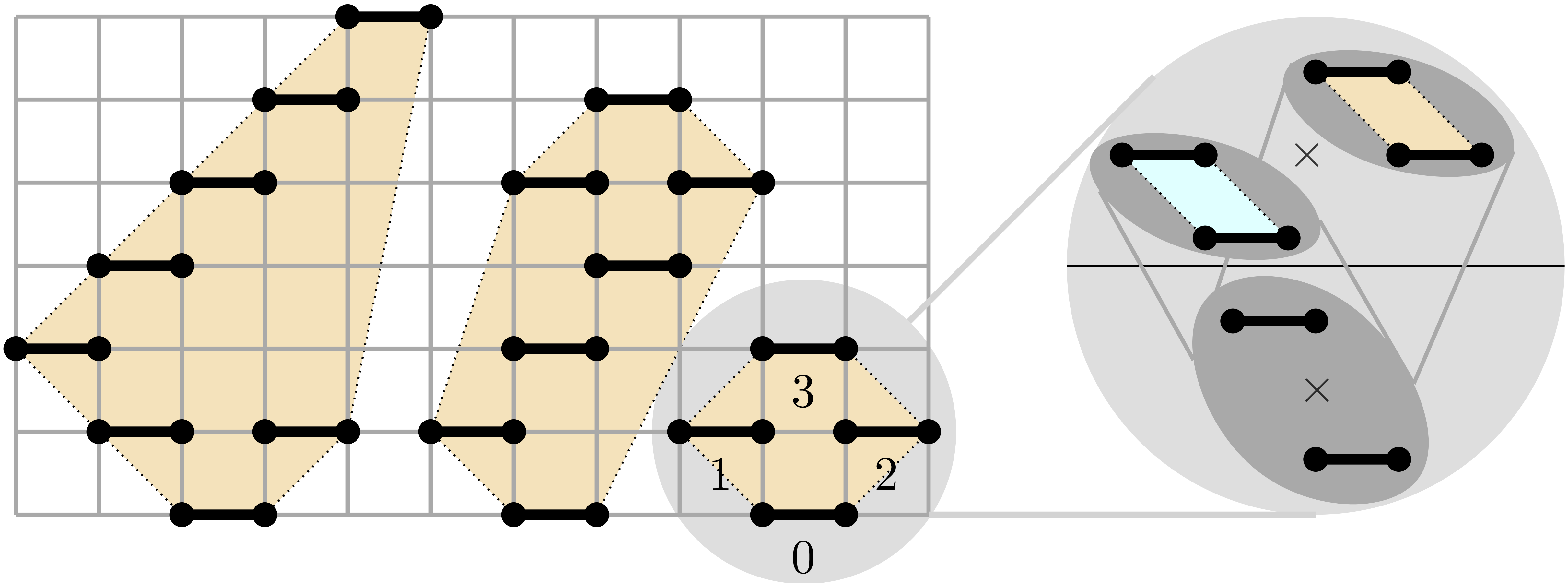

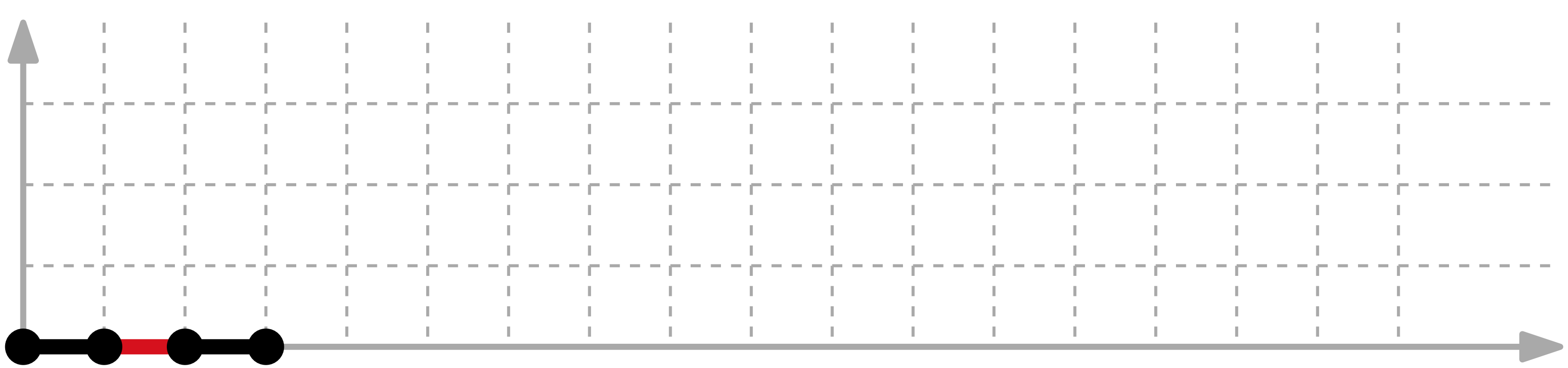

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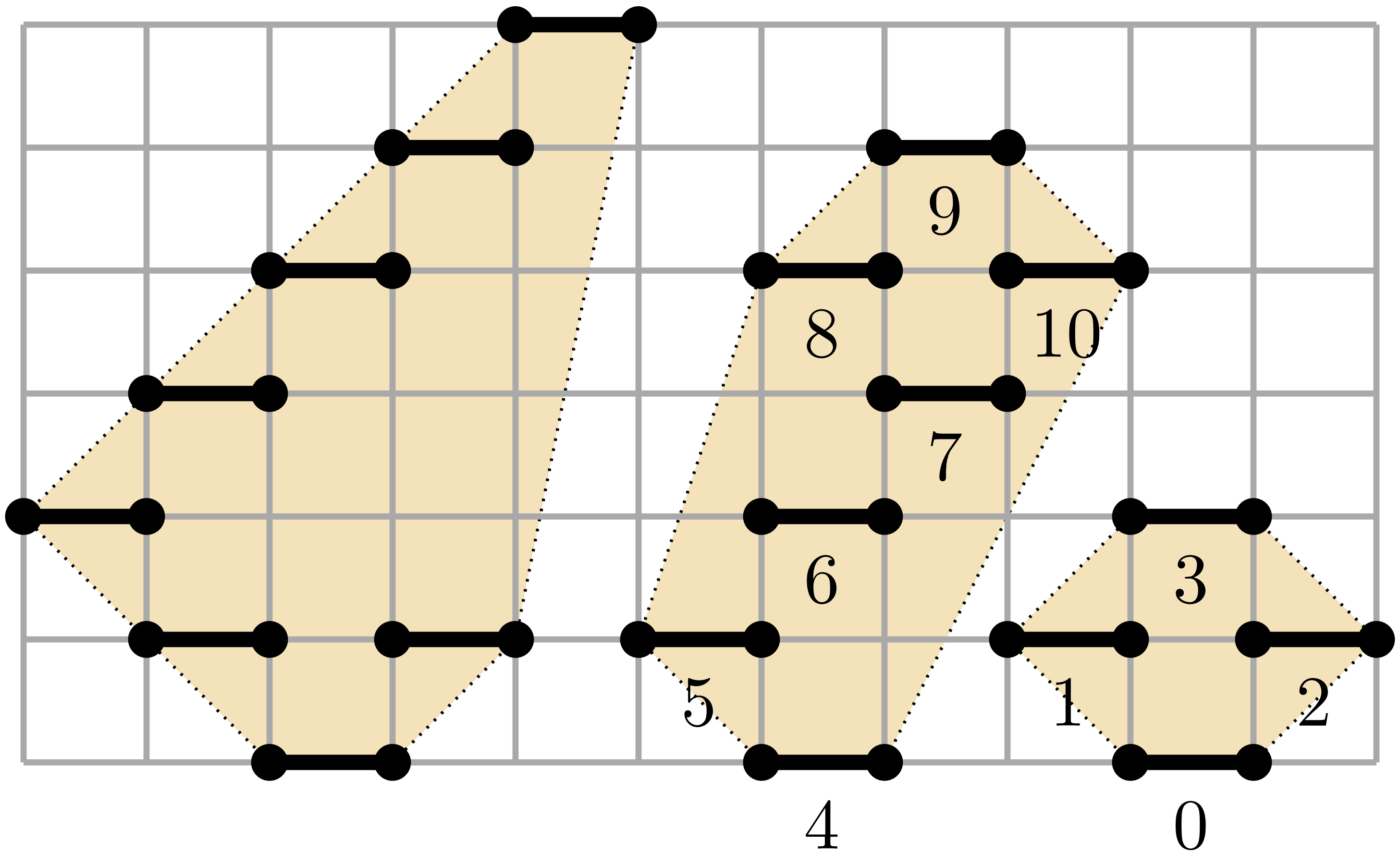

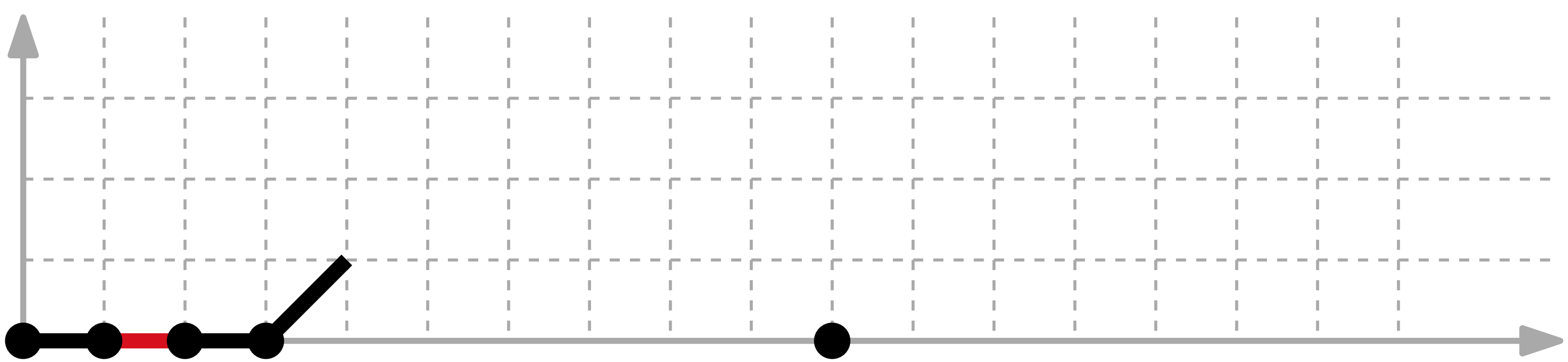

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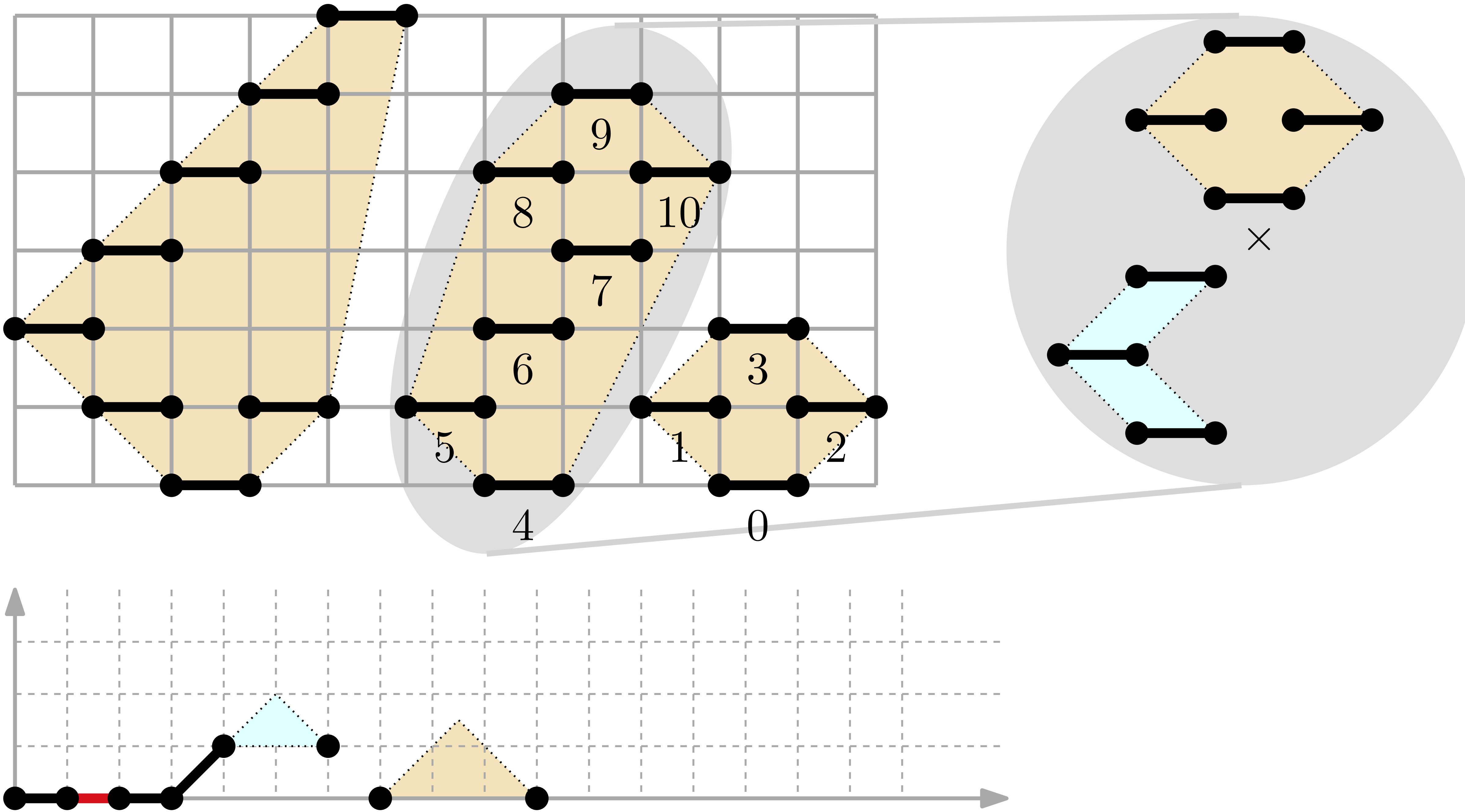

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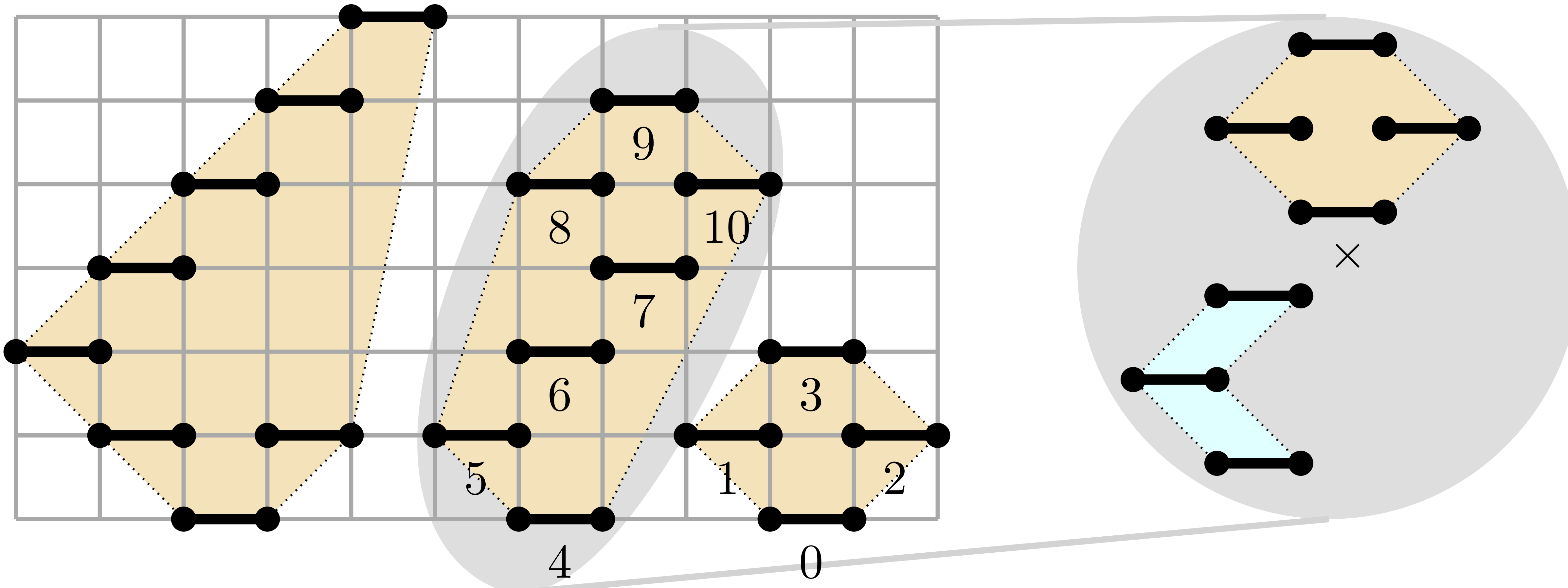

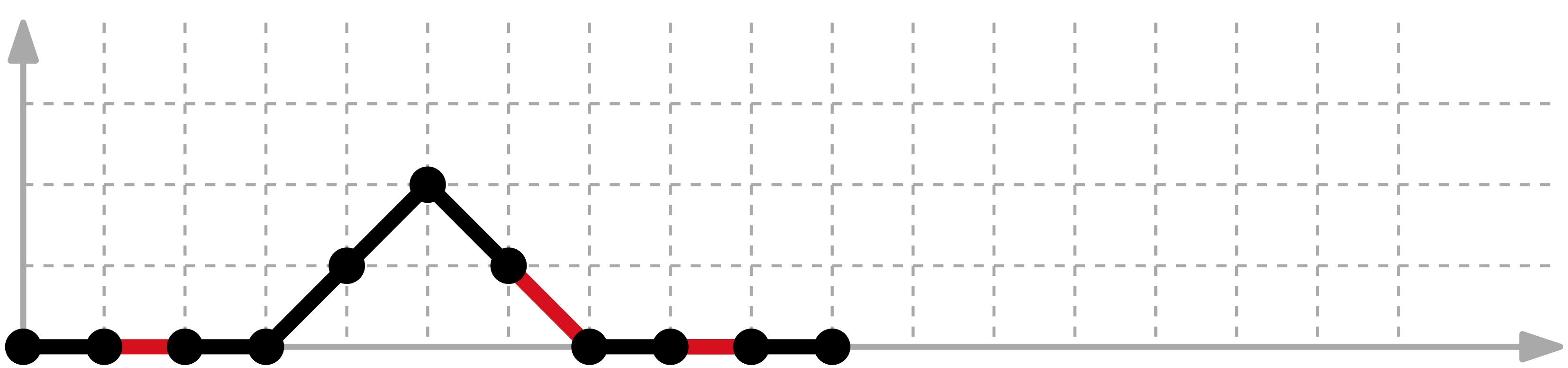

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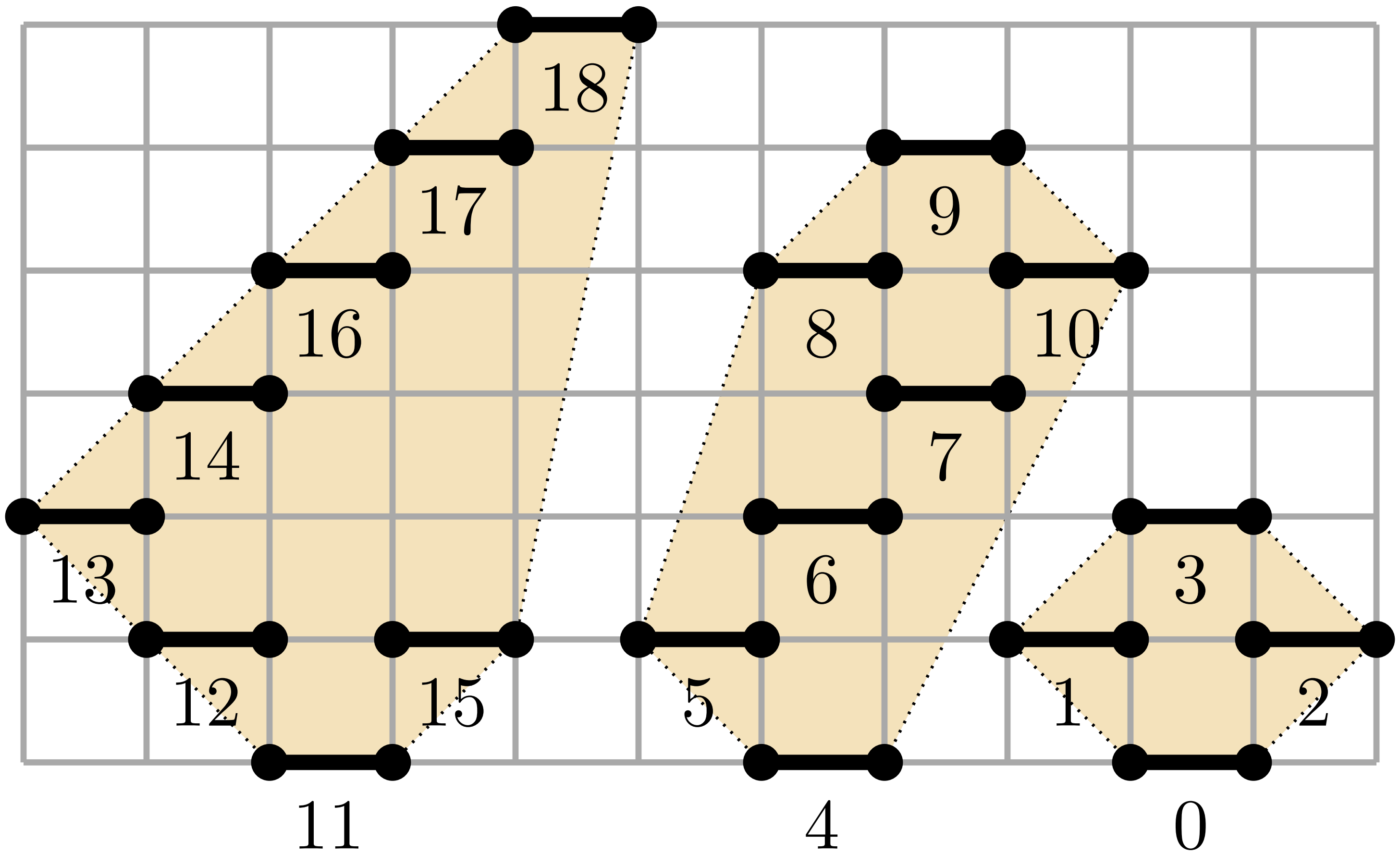

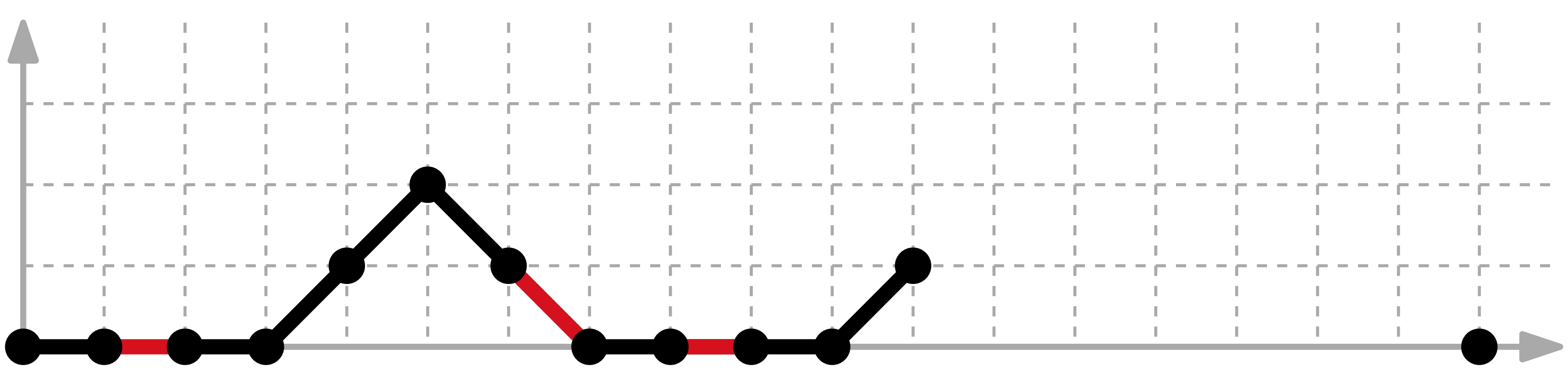

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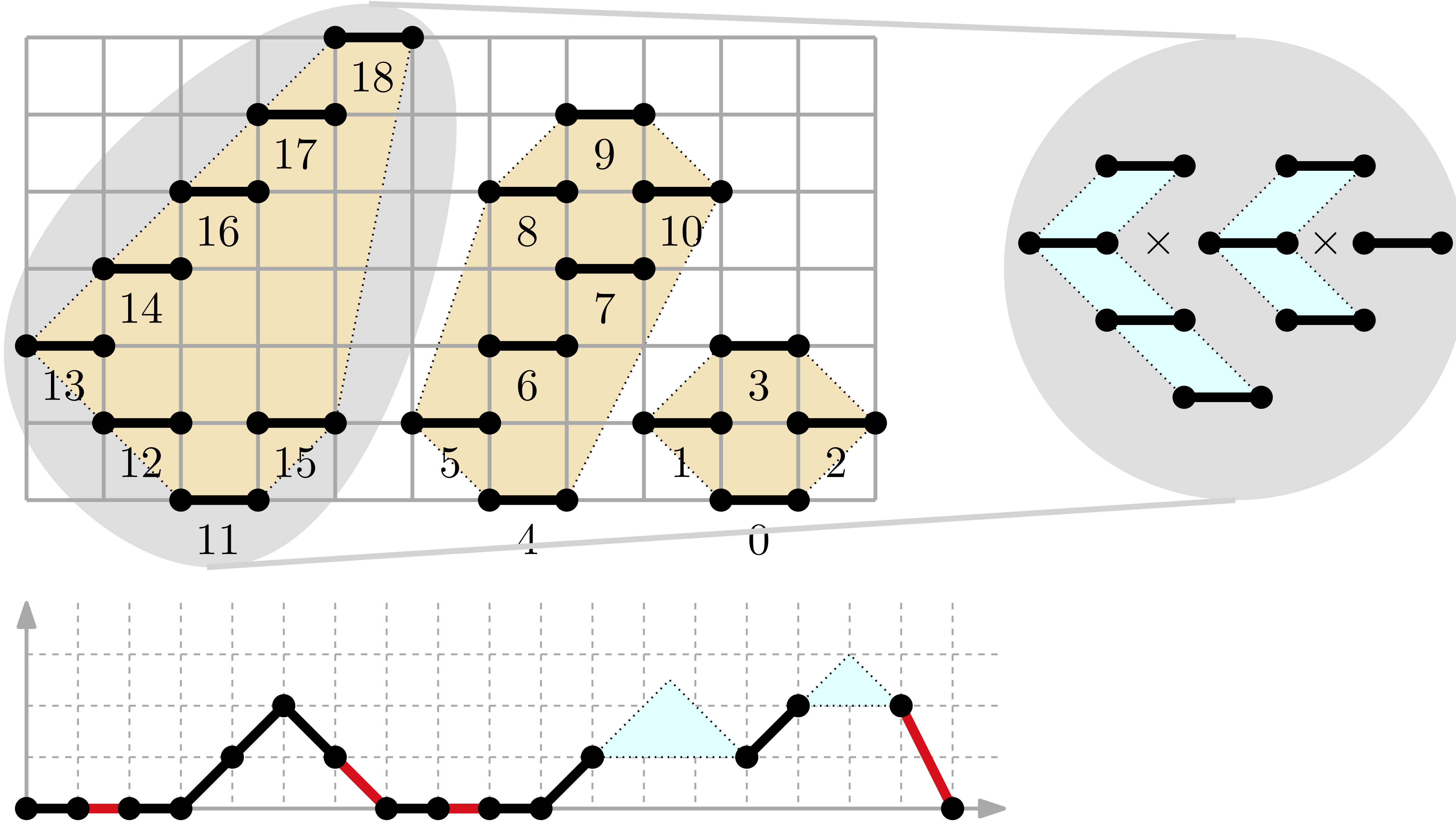

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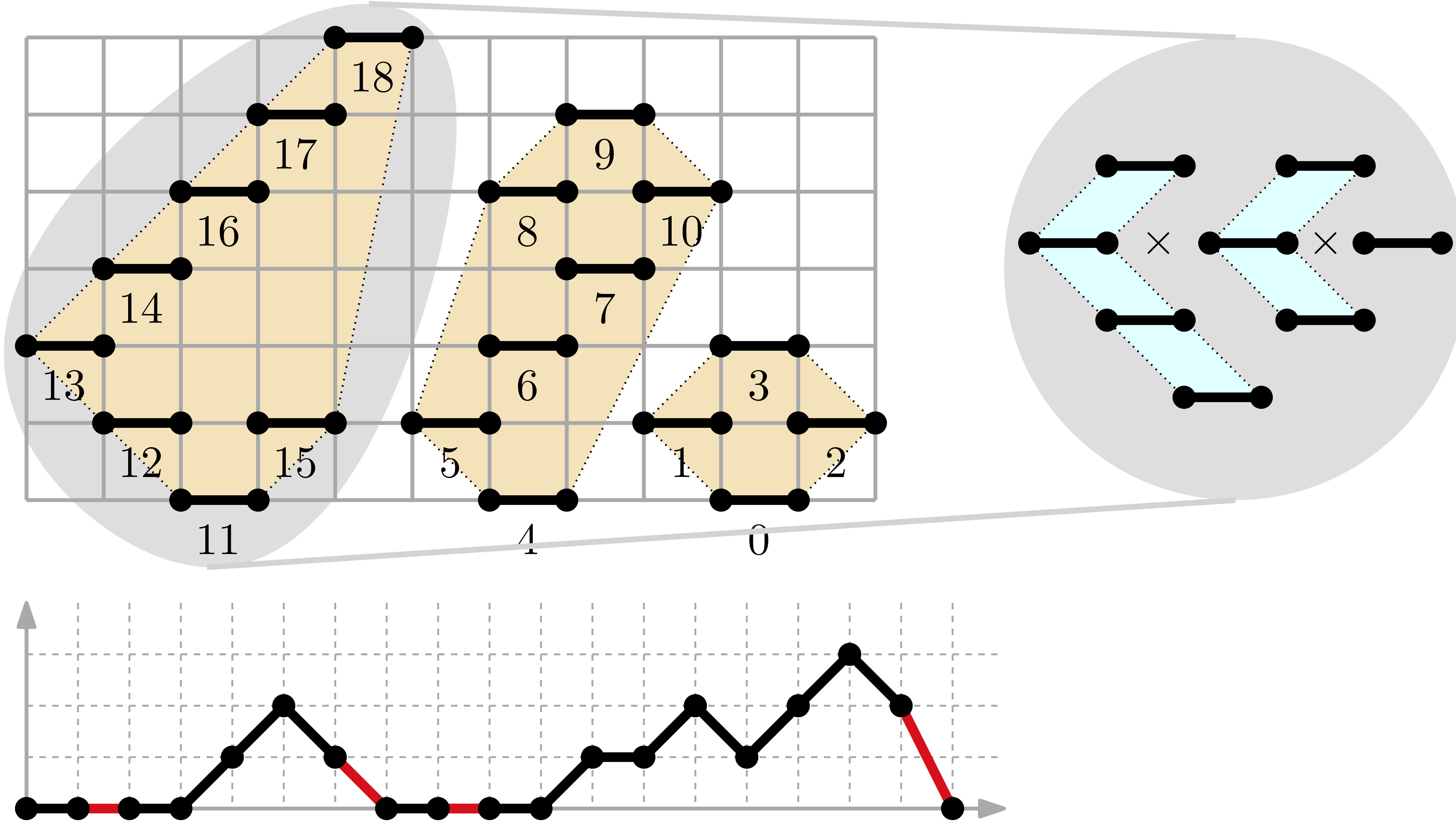
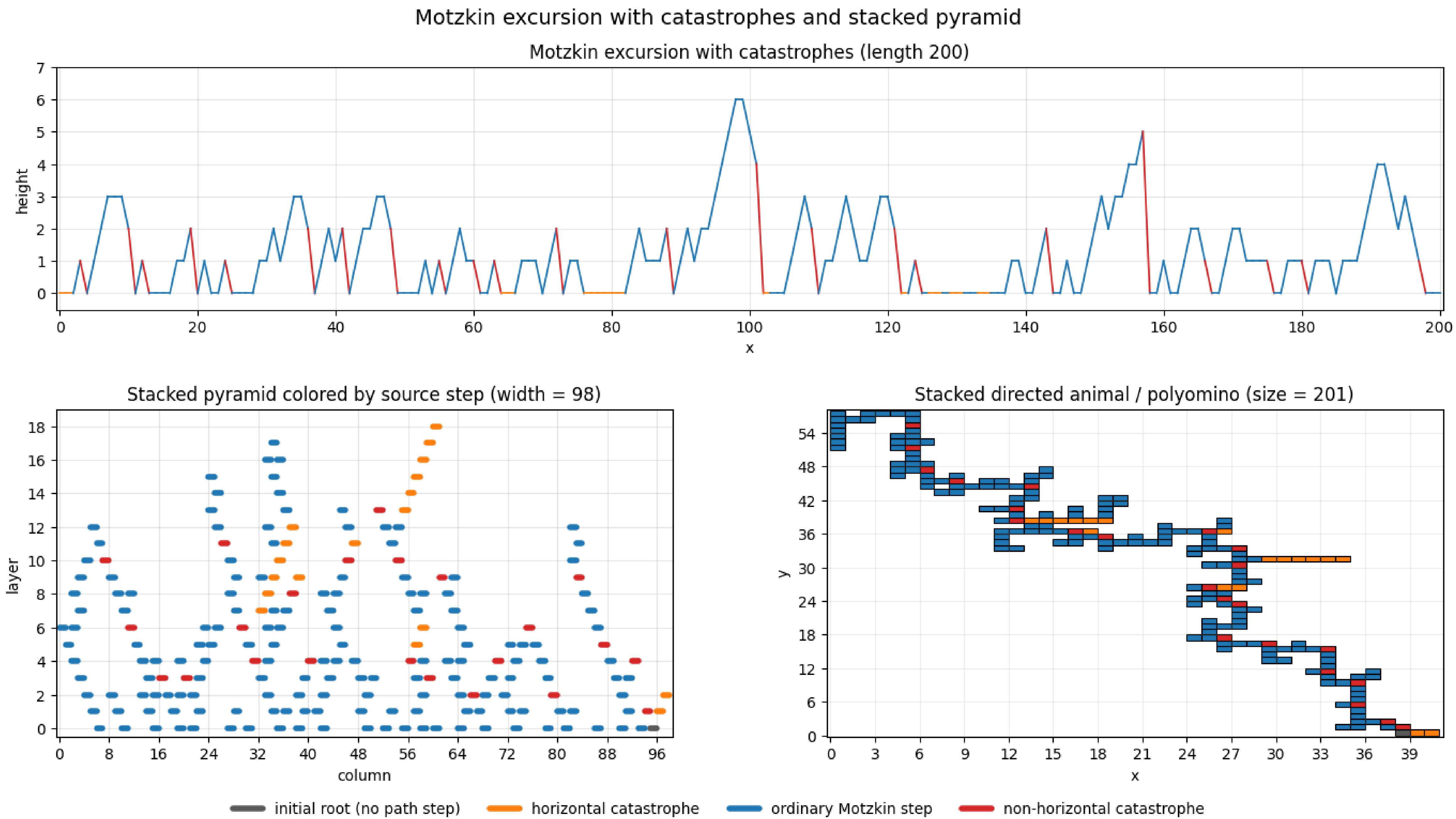


How does an average stacked directed animal look like?



Using the bijection

Polyominoes		Lattice paths
number of minimal dimers	$\longleftrightarrow$	number of "real" catastrophes
distance between minimal dimers	$\longleftrightarrow$	size of catastrophe

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Polyominoes		Lattice paths
number of minimal dimers	$\longleftrightarrow$	number of "real" catastrophes
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\# catastrophes

$\cong$

$3/28 \cdot n$

$\leq$ width

$= 3/28 \cdot n$

[Bousquet-Mélou & Rechnitzer, 2002]

Using the bijection

Polyominoes		Lattice paths
number of minimal dimers	$\longleftrightarrow$	number of "real" catastrophes
distance between minimal dimers	$\longleftrightarrow$	size of catastrophe

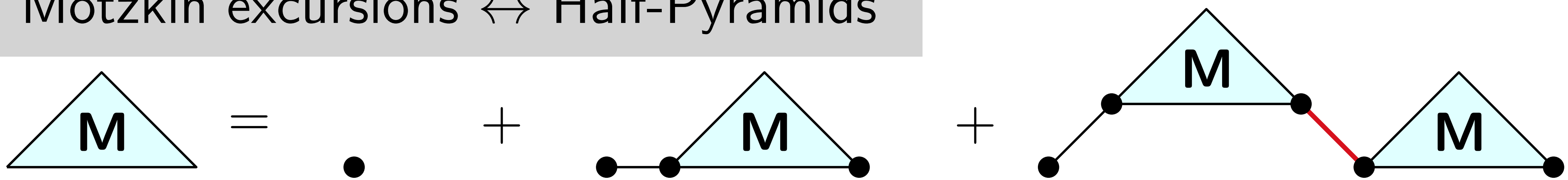
$$\begin{array}{ccccc} \text{\# catastrophes} & + & \boxed{\text{cum. size of all catastrophes}} & \leq & \text{width} \\ \cong & & \cong & & \\ 3/28 \cdot n & + & 6/28 \cdot n & = & \cancel{3/28 \cdot n} \\ & & & & 9/28 \cdot n \end{array}$$

[Bousquet-Mélou & Rechnitzer, 2002]

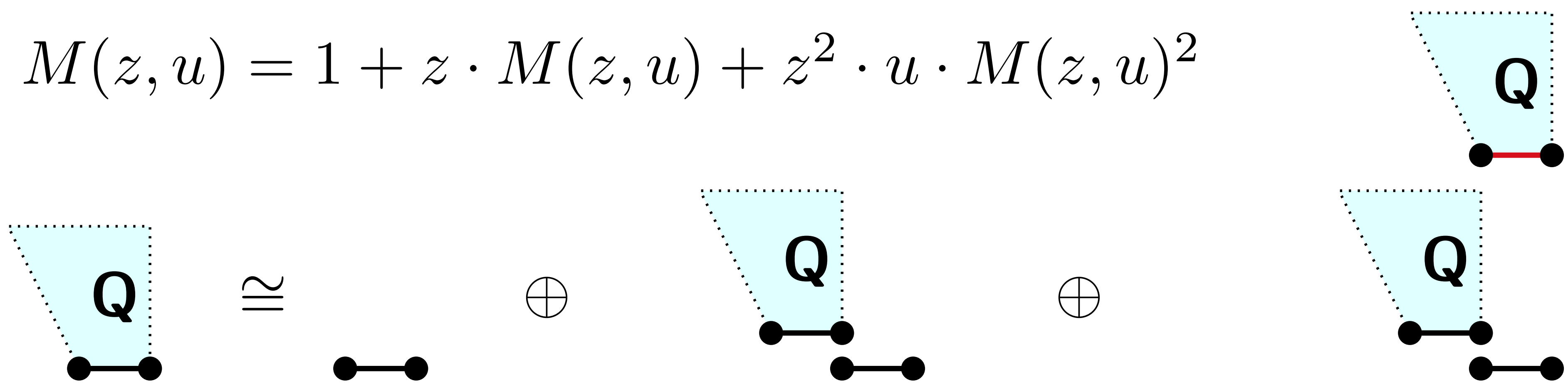
Improved upper bounds 1

Idea: Mark steps that do not contribute to the width

Motzkin excursions ↔ Half-Pyramids



$$M(z,u) = 1 + z \cdot M(z,u) + z^2 \cdot u \cdot M(z,u)^2$$



Improved upper bounds 1

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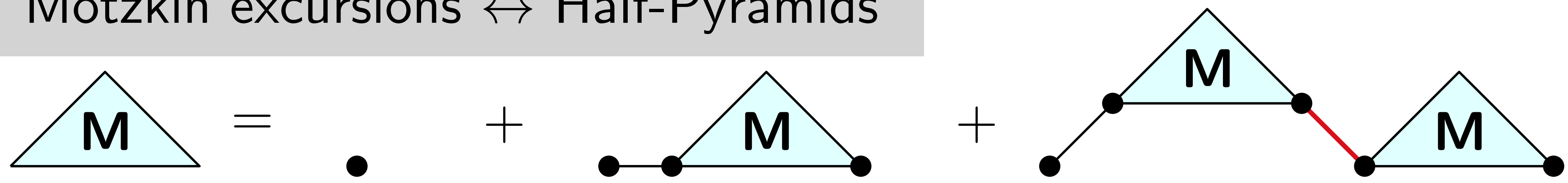
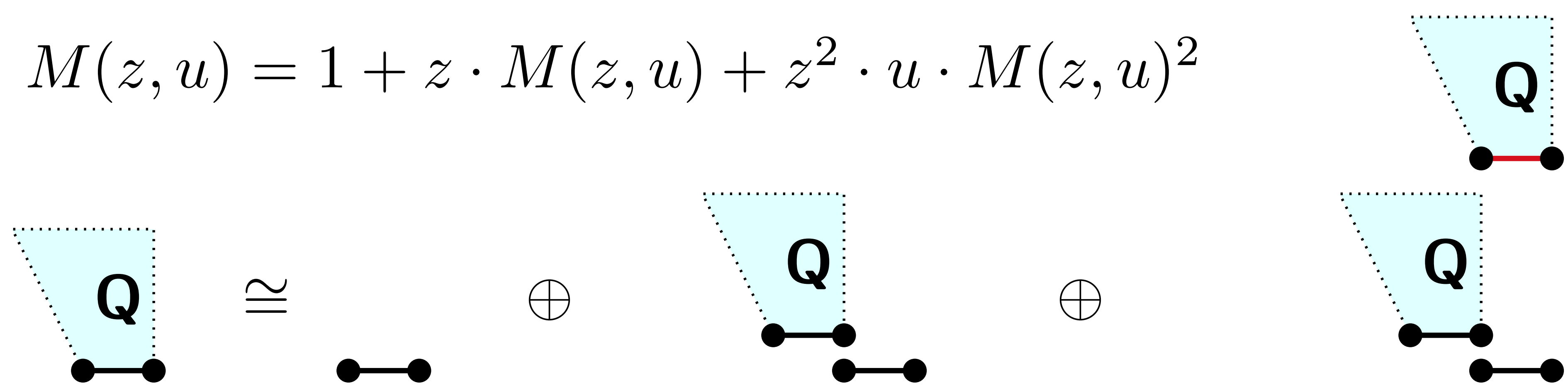
Motzkin excursions $\leftrightarrow$ Half-Pyramids



$$M(z,u) = 1 + z \cdot M(z,u) + z^2 \cdot u \cdot M(z,u)^2$$

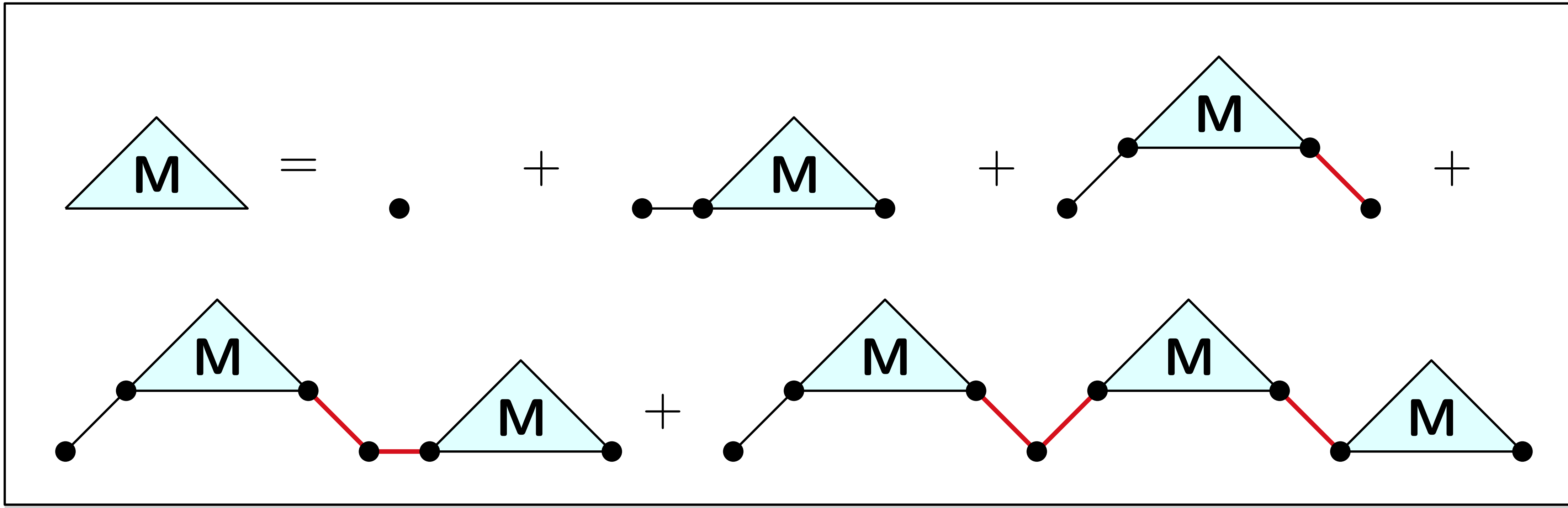


$\longrightarrow$ asymptotic number of marked dimers in a half-pyramid $\sim n/3$

$\longrightarrow$ asymptotic width of a stacked pyramids $\leq 6/7 \cdot n$

Improved upper bounds 2

Idea: Expand the recursive decomposition

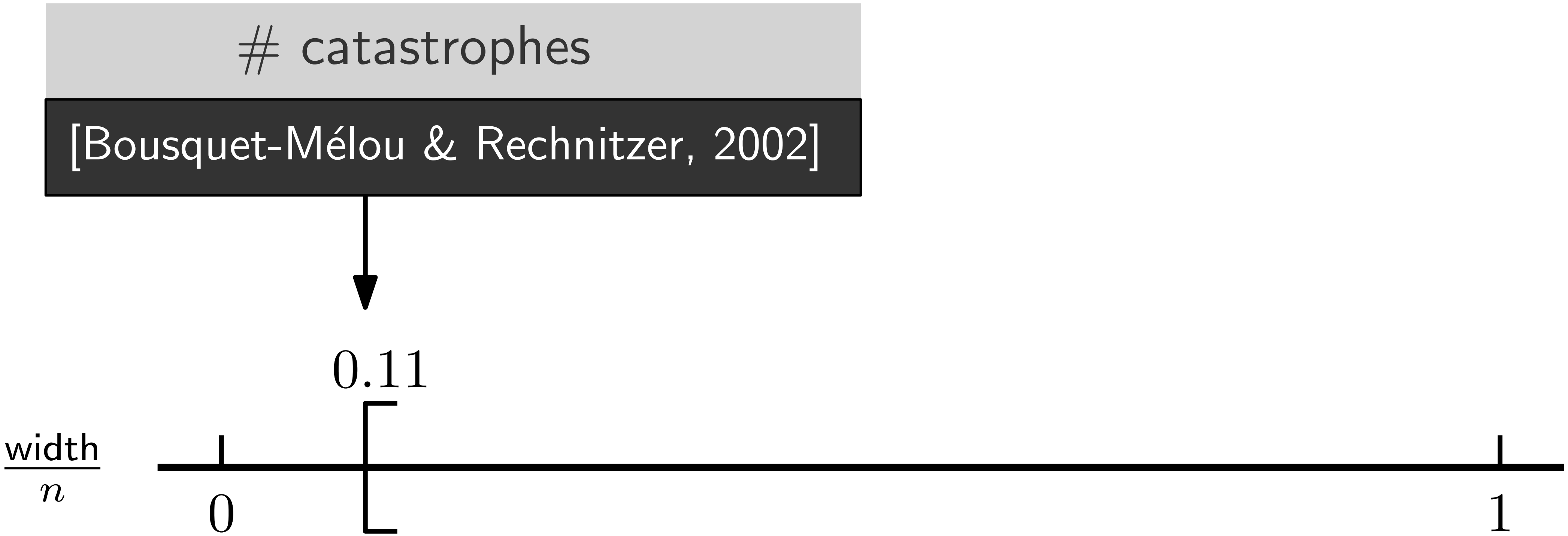


$$M = z + z \cdot M + z^2 \cdot u \cdot M + z^3 \cdot u^2 \cdot M^2 + z^4 \cdot u^3 \cdot M^3$$

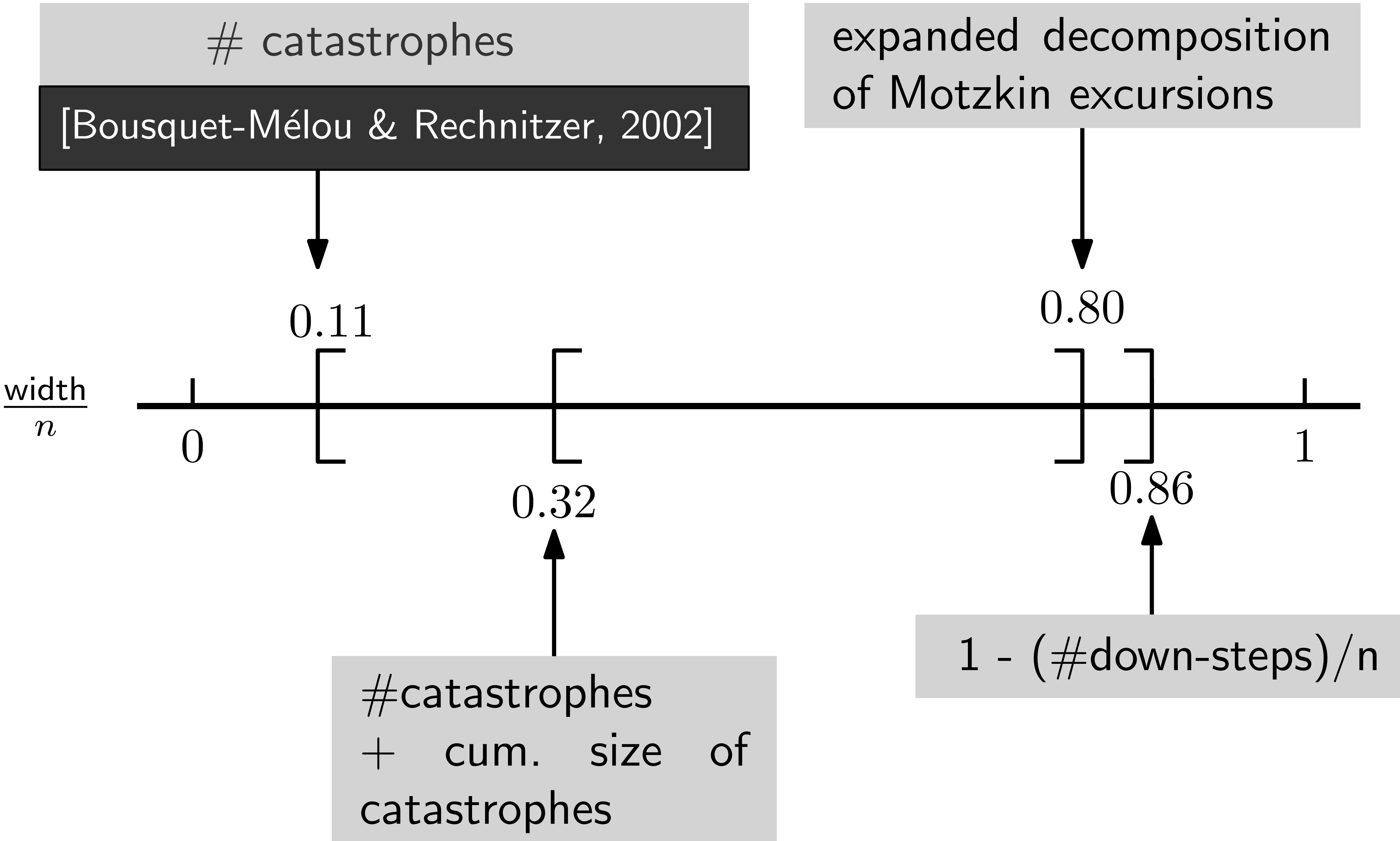
$\longrightarrow$ asymptotic number of marked dimers in a half-pyramid $\sim n/2$

$\longrightarrow$ asymptotic width of a stacked pyramids $\leq 45/56 \cdot n$

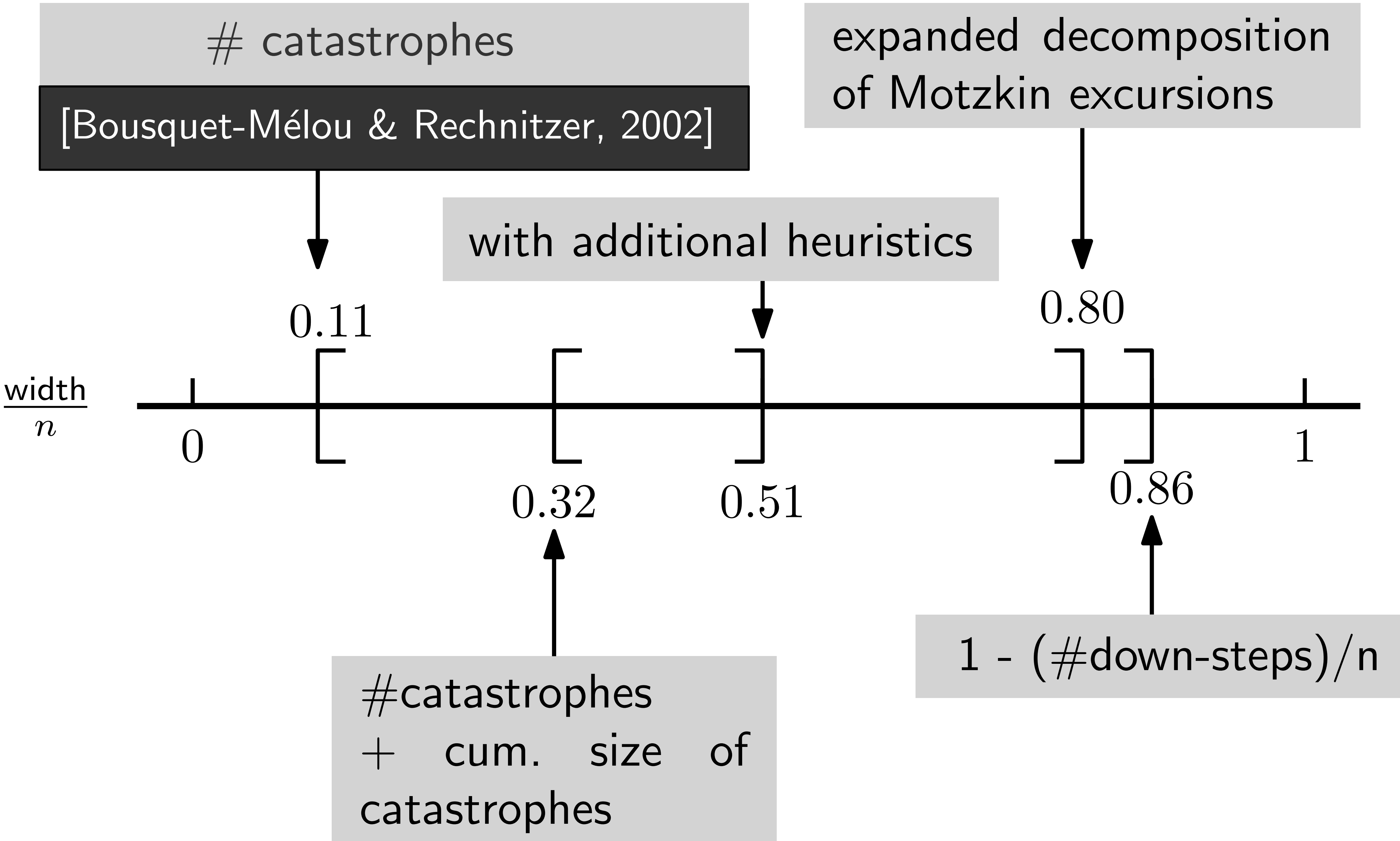
Improved bounds. Overview



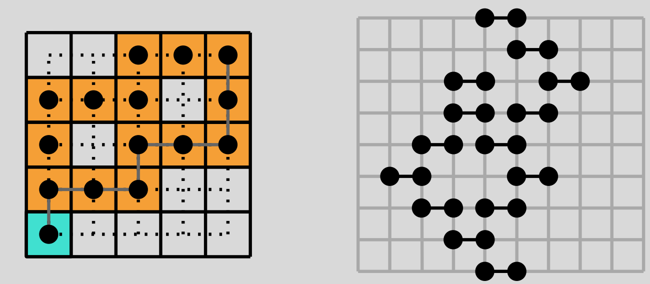
Improved bounds. Overview



Improved bounds. Overview



Summary

Background	Surprising connection between lattice paths with catastrophes and a subclass of lattice animals	

Summary

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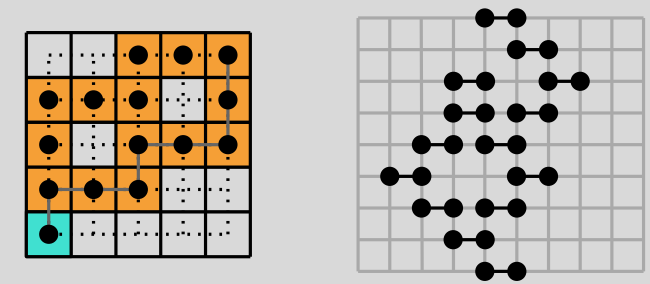
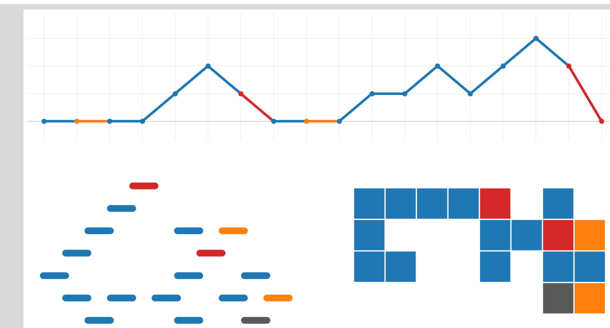
Surprising connection between lattice paths with catastrophes and a subclass of lattice animals



Contribution

explicit bijection between

- Motzkin excursions with catastrophes
- stacked directed animals



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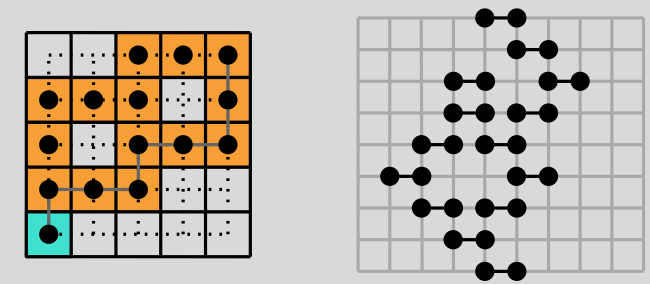

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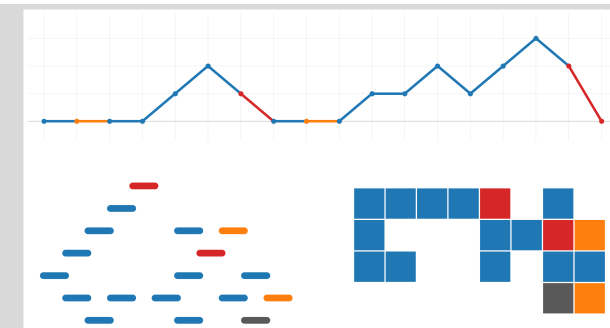

Application

study lattice path parameters to derive improved upper and lower bounds on the width of stacked pyramids