Branching random walks: speed and entropy

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- **BRW: Branching random walks**
- **Empirical distributions**
- **Speed and displacement**
- **Entropy**

Joint research with

Robin Kaiser

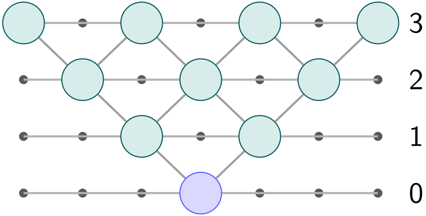
Jérémie Brieussel

Martin Klötzer

Branching random walks BRW

BRW combines two mechanisms:

1. each particle branches according to π;
2. each child does a random walk on G.



State space: infinite graph G.
For this talk the state space is regular tree $\mathbb{T}_d$ (or $\mathbb{Z}^d$).

Branching process

Generation 0

- Start with one ancestor and offspring distribution π.

Branching process

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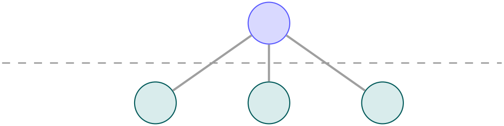
Generation 1



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- Produces k children with distribution $\pi(k)$ and then it dies.

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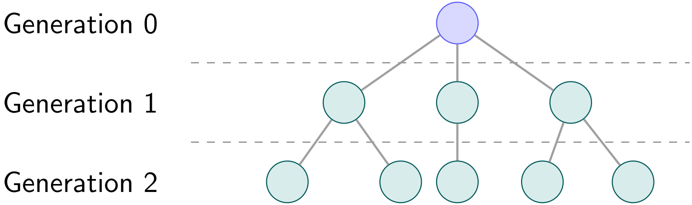

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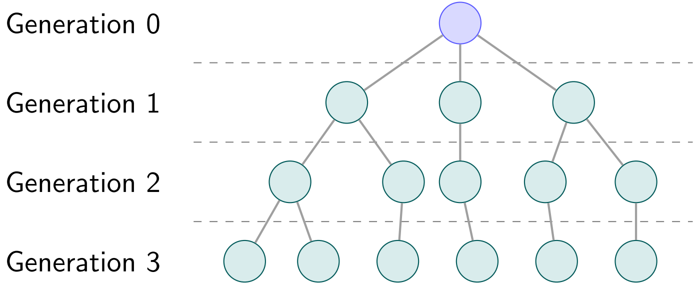

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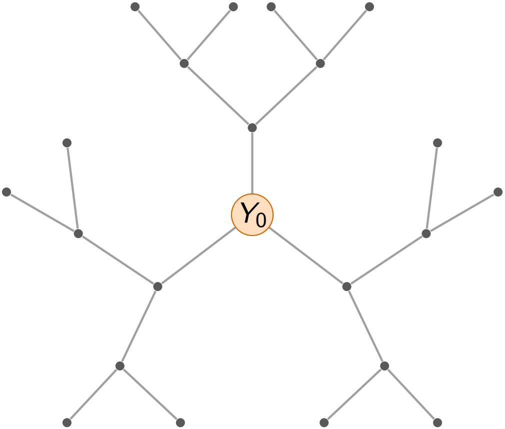
- Produces k children with distribution $\pi(k)$ and then it dies.
- Each new particle branches independently and then it dies.
- The genealogy is built generation by generation.
- Z_n = number of particles in the n-th generation.

Random walk on a tree $\mathbb{T}_d$



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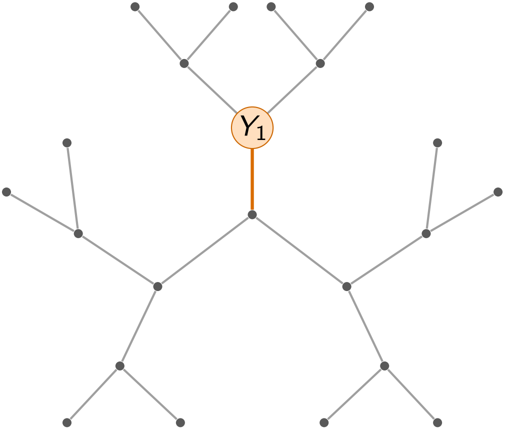

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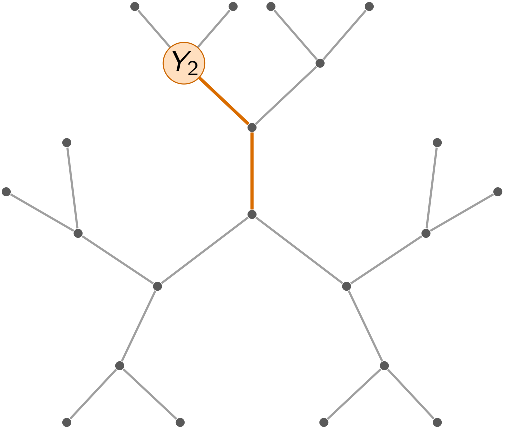

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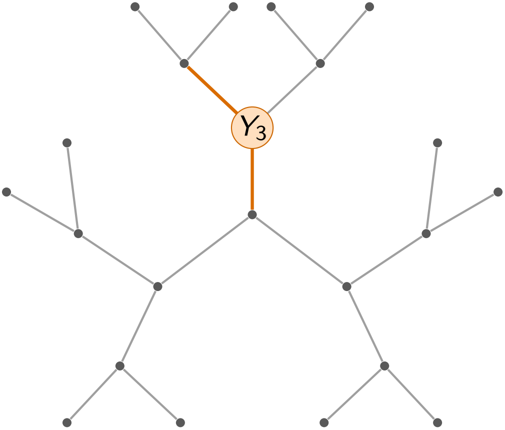

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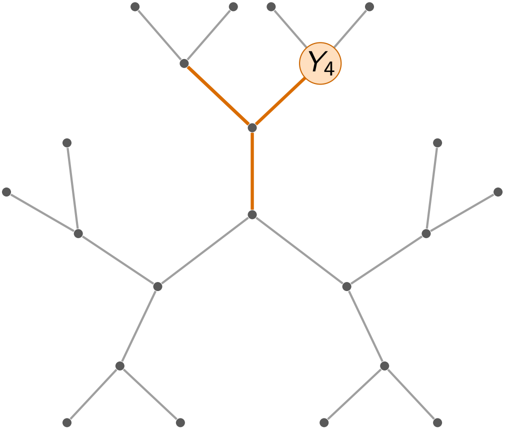

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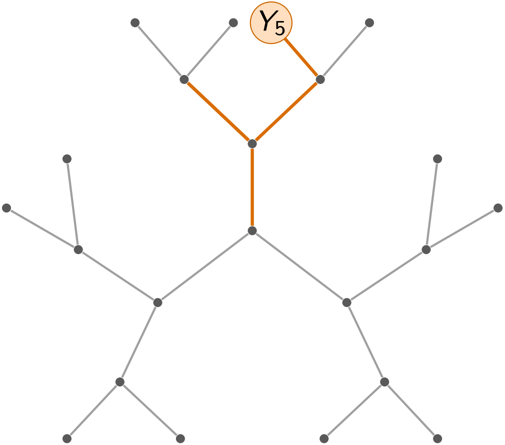

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$$\begin{cases} \rho > 1 & \text{supercritical} \\ \mathbb{E}(L \log(L)) < \infty, \quad L \sim \pi & L \log L \text{ condition} \end{cases}$$

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- Kesten-Stigum [1966]

$$\frac{Z_n}{\rho^n} \to W \text{ almost surely and } \mathbb{P}[W > 0] = 1.$$

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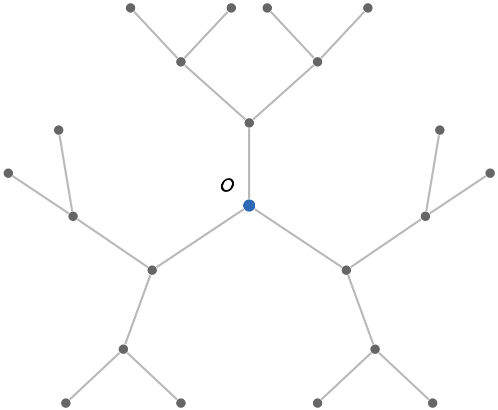
For $n \geq 0$ let $M_n : V \to \mathbb{N}$ random population at time n; for $x \in V$

$$M_n(x) := \# \text{ individuals at vertex } x \text{ at time } n$$

$$M_n = \sum_{x \in V} M_n(x) \delta_x \qquad \text{measure on G}$$

BRW: $(M_n)_{n\in\mathbb{N}}$ sequence of random populations



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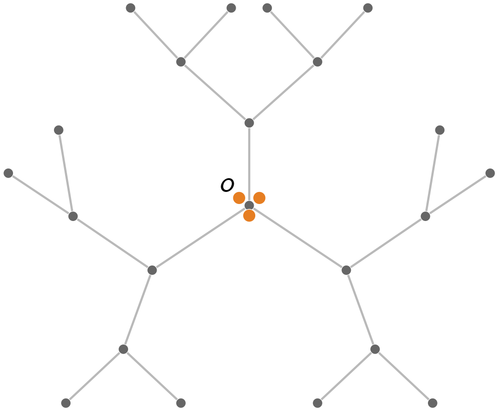

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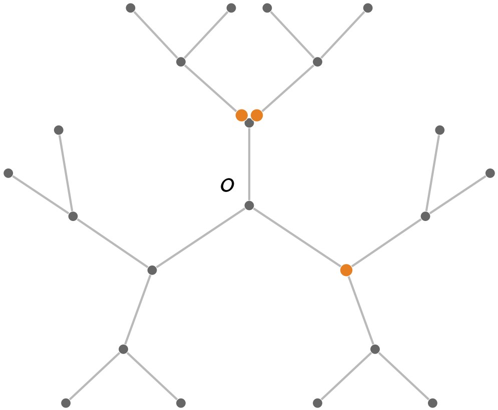

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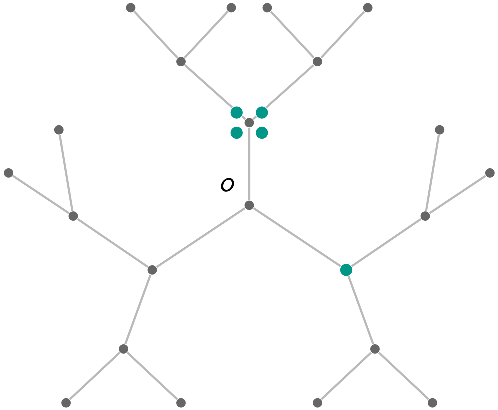

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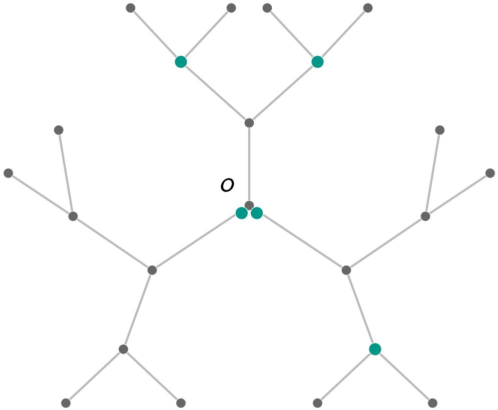

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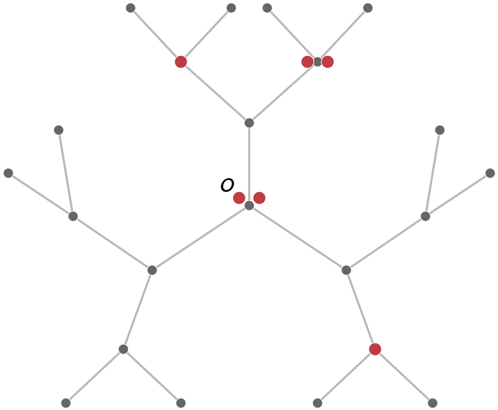

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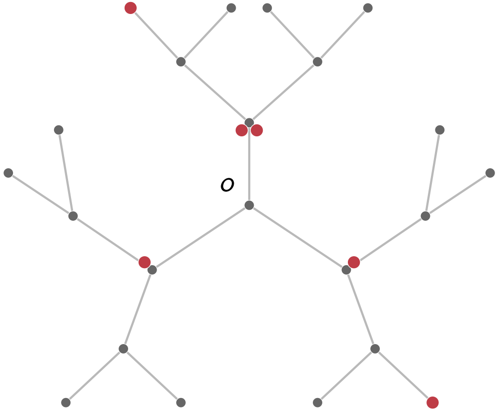

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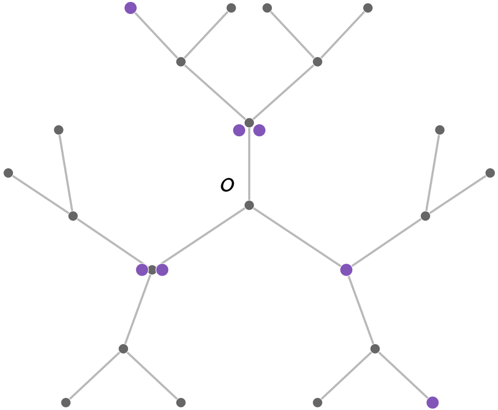

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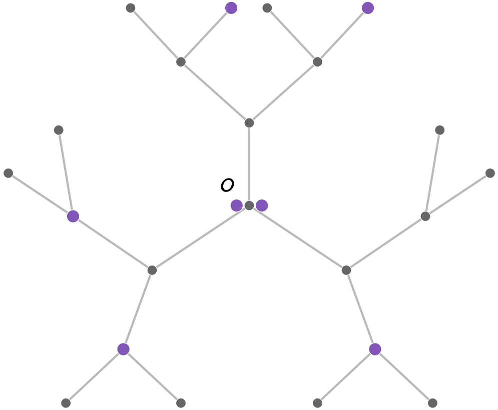

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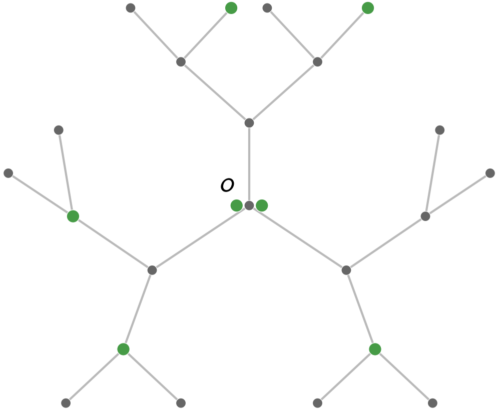

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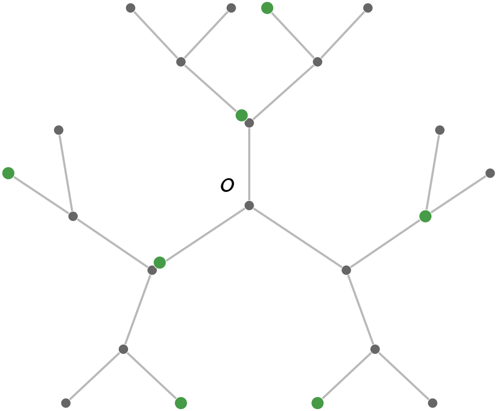

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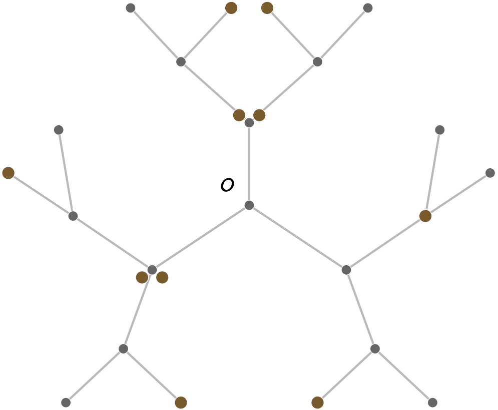

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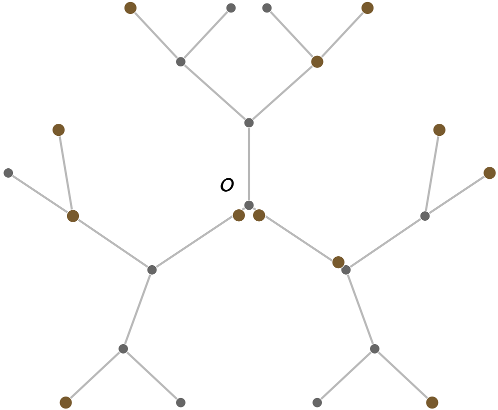

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  - Transience: every finite subset of G visited finitely many times a.s. by the BRW.
  - Recurrence: every finite subset of G visited ∞ often a.s.

Phase transitions for BRW

Transience/recurrence phase transition for BRW occurs at $1/r$: Benjamini-Peres (1996), Gantert-Müller (2007)

- $\rho \leq \frac{1}{r}$: BRW is transient (extinction if $\rho \leq 1$ and weak survival if $\rho \in (1, \frac{1}{r}]$).
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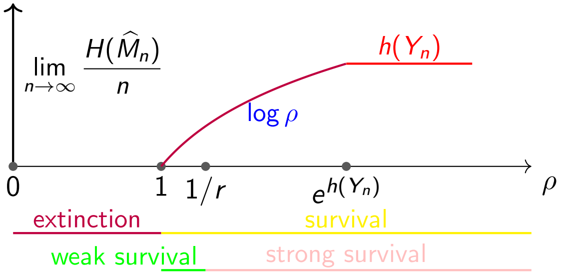
Aim of this talk: existence of a third phase transition!



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Theorem: [Candellero-Hutchcroft,Kaimanovich-Woess 2023]

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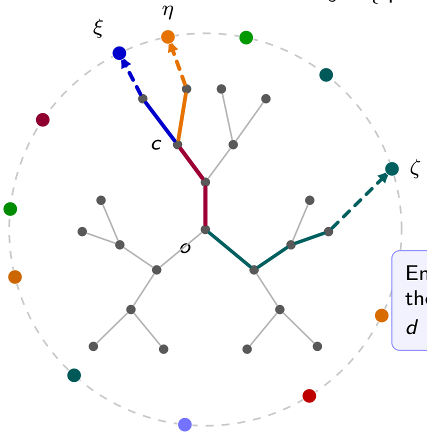
Theorem: [Candellero-Hutchcroft,Kaimanovich-Woess 2023]

Let (M_n) be supercritical BRW on G with $L \log L$ condition and transient underlying random walk. Then, almost surely (for almost all realizations of BRW)

$$\widehat{M}_n = \frac{M_n}{Z_n} \text{ converges weakly to a } \underline{\text{random measure}} \text{ on } \partial G.$$

End compactification of $\mathbb{T}_3$

$\partial\mathbb{T}_3$ = {space of ends}={infinite rays}



Ends ξ and η agree on $[0, c]$ then split: $d(\xi, \eta) = 2^{-|c|}$. d metric on $\widehat{\mathbb{T}}_3$

$\widehat{\mathbb{T}}_3 \;=\; \mathbb{T}_3 \cup \partial\mathbb{T}_3$: vertices plus infinite rays.
$w = [w_0, w_1, \ldots] \to \xi$ if length common initial segment $\to \infty$.

Quantitative behaviour of empirical distributions

$\widehat{M}_n = \frac{M_n}{Z_n}$ empirical distributions of the BRW on G.

Questions (Kaimanovich-Woess [2023])

Empirical versions of limit theorems for BRW:

- Speed: random walk has speed ℓ, does the whole cloud of particle move with the same speed?
- CLT for $\widehat{M}_n$: if speed of a random walk has Gaussian fluctuations, does the BRW cloud inherit the fluctuations?
- Entropy: If we know the entropy of the underlying random walk, what can we say about the entropy of $\widehat{M}_n$?

Average speed and CLT for the empirical distributions

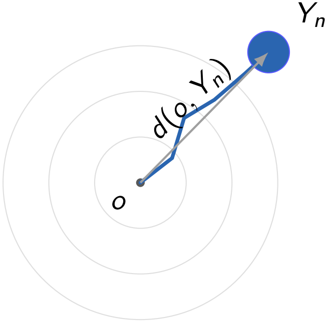
Underlying random walk: Let $(Y_n)_{n\geq 0}$ be a transient random walk on G started at o.

Rate of escape/Speed

$$\ell = \lim_{n\to\infty} \frac{d(o, Y_n)}{n}.$$

For the CLT part, assume that for some $\sigma > 0$,

$$\frac{d(o, Y_n) - \ell n}{\sigma\sqrt{n}} \xrightarrow{n\to\infty} \mathcal{N}(0,1).$$



the distance from o grows linearly $\asymp \ell n$, with possible Gaussian fluctuations

Speed of the empirical cloud of particles

Define the normalized average speed (displacement)

$$D_n := \frac{1}{n}\int d(o,x)\, d\widehat{M}_n(x) = \frac{1}{nZ_n}\sum_{x\in G} d(o,x)M_n(x).$$

Theorem: [Kaiser-Klötzer-Sava-Huss 2026]

Empirical rate of escape on transitive graphs G: if random walk (Y_n) has speed $\ell = \lim_{n\to\infty}\frac{d(o,Y_n)}{n}$ (almost sure limit), and offspring distribution with $\pi(0)=0$, $\rho>1$, and finite second moment, then

$$D_n \xrightarrow[n\to\infty]{a.s.} \ell.$$

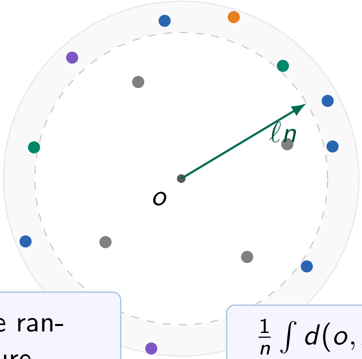
Equivalently, on the survival event where $W>0$,

$$\frac{1}{n\rho^n}\sum_{x\in G} d(o,x)M_n(x) \xrightarrow[n\to\infty]{a.s.} \ell W.$$

Picture of the speed theorem

BRW cloud travels at the same linear speed as one random walk; random factor is the martingale limit W when normalizing by ρ^n.

typical radial shell $d(o,x) \approx \ell n$



$\widehat{M}_n$ is a finite random measure.

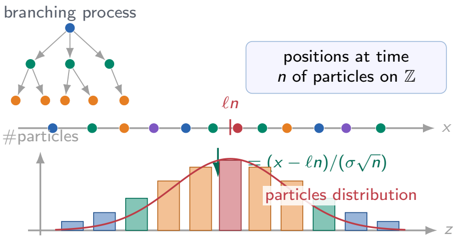
$\frac{1}{n}\int d(o,x)\, d\widehat{M}_n(x) \to \ell.$

Stam-type CLT on $\mathbb{Z}$

Stam [1966], Biggins [1988]

BRW on $\mathbb{Z}$ with speed ℓ and variance σ^2, for each $y \in \mathbb{R}$,
$$\frac{1}{\rho^n} Z_n\Big((-\infty, \ell n + \sigma\sqrt{n}\, y]\Big) \longrightarrow W\Phi(y)$$

- center at the drift ℓn;
- rescale $\sqrt{n}$;
- the $\#$ branching particles in $(-\infty, \ell n + \sigma\sqrt{n}\, y]$ rescaled by ρ^n becomes Gausssian.

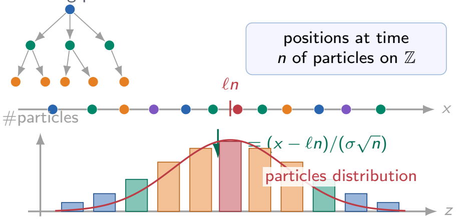


Stam-type CLT on $\mathbb{Z}$

Stam [1966], Biggins [1988]

BRW on $\mathbb{Z}$ with speed ℓ and variance σ^2, for each $y \in \mathbb{R}$,

$$\frac{1}{\rho^n} Z_n\Big((-\infty, \ell n + \sigma\sqrt{n}\, y]\Big) \longrightarrow W\Phi(y)$$



- center at the drift ℓn;
- rescale $\sqrt{n}$;
- the # branching particles in $(-\infty, \ell n + \sigma\sqrt{n}\, y]$ rescaled by ρ^n becomes Gausssian.

CLT for empirical distributions $\widehat{M}_n = \frac{M_n}{Z_n}$ of BRW on $\mathbb{T}_d$?

What about the set of particles in $\mathbb{T}_d$ in BRW that deviate at most of order $\sqrt{n}$ from the average speed ℓn?

$$A_n(a, b) = \left\{ x \in \mathsf{G} : a \leq \frac{d(o, x) - \ell n}{\sigma\sqrt{n}} \leq b \right\}.$$

CLT for the empirical distributions $\widehat{M}_n = \frac{M_n}{Z_n}$

For $a < b$, define the radial CLT window

$$A_n(a,b) = \left\{ x \in \mathsf{G} : a \leq \frac{d(o,x) - \ell n}{\sigma \sqrt{n}} \leq b \right\}.$$

Theorem: [Kaiser-Klötzer-Sava-Huss 2026]

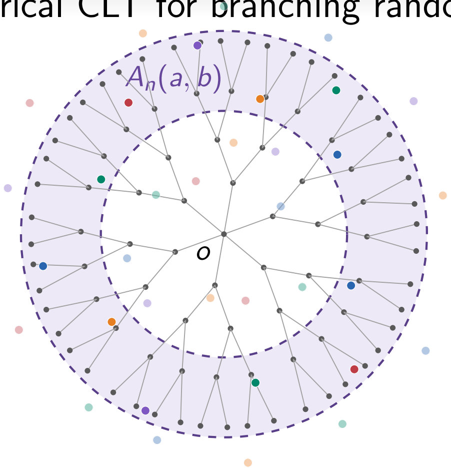
Empirical CLT for BRW: If the underlying random walk (Y_n) satisfies a CLT (and $\rho > 1$ and finite second moment for π)

$$\frac{d(o,Y_n) - \ell n}{\sigma \sqrt{n}} \xrightarrow[n\to\infty]{} \mathcal{N}(0,1),$$

then the empirical rescaled radial empirical distributions converge weakly in probability to $W\mathcal{N}(0,1)$:

$$\frac{1}{\rho^n} M_n\big(A_n(a,b)\big) \xrightarrow[n\to\infty]{\mathbb{P}} W\big(\Phi(b) - \Phi(a)\big).$$

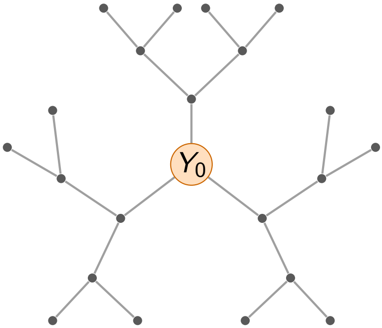
Empirical CLT for branching random walks



Shell between $r_a = \ell n + \sigma\sqrt{n}\, a$ and $r_b = \ell n + \sigma\sqrt{n}\, b$.

Many particles lie outside the shell; only particles in the shell contribute to $\widehat{M}_n\big(A_n(a,b)\big)$.

Asymptotic entropy $h(Y_n)$ of random walk Y_n



- Entropy: how many vertices are plausible locations of Y_n?

Meaning of $h(Y_n)$

$e^{nh(Y_n)}$ is the exponential growth rate of the part of the tree explored by the walk.

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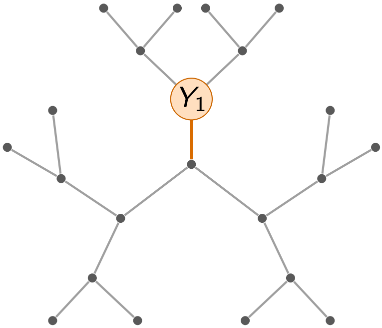

- Entropy: how many vertices are plausible locations of Y_n?
- For $p_n(x) = \mathbb{P}(Y_n = x)$, (Shannon) entropy of Y_n is

$$H(Y_n) = -\sum_{x \in \mathbb{T}_d} p_n(x) \log p_n(x).$$

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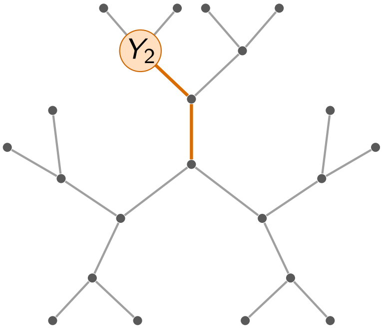

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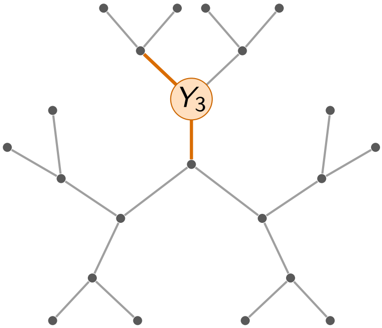

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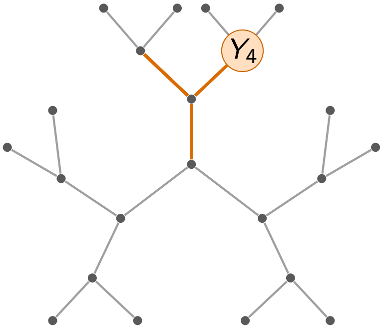

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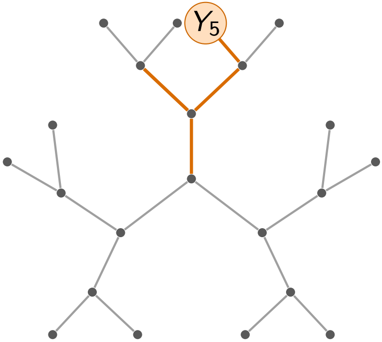

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Example: simple walk on the 3-regular tree

For the simple random walk on $\mathbb{T}_3$, the distance from the root behaves like a biased walk on $\mathbb{N}$.

Speed

Away from the root, one edge goes back and two edges go forward. Hence the distance has drift

$$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}, \qquad d(0, Y_n) \approx \frac{n}{3}.$$

Asymptotic entropy

The ball of radius r has about 2^r vertices, so typical sites at time n occupy roughly

$$2^{n/3} = e^{(n/3)\log 2}$$

positions. Thus

$$h(Y_n) = \frac{1}{3} \log 2.$$

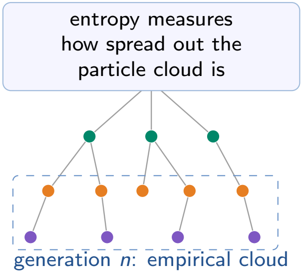
Entropy of the empirical distributions

For a branching random walk on G, the empirical distribution at time n is

$$\widehat{M}_n(x) = \frac{M_n(x)}{Z_n}.$$

Its Shannon entropy is

$$H(\widehat{M}_n) = -\sum_{x \in G} \widehat{M}_n(x) \log \widehat{M}_n(x).$$



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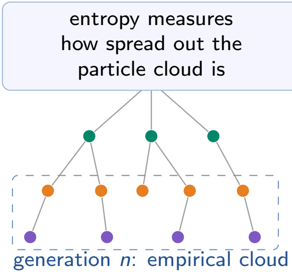


Question

Does the limit

$$\lim_{n\to\infty} \frac{H(\widehat{M}_n)}{n}$$

exist almost surely, and what is its value?

Phase transition for the entropy

Theorem: [Brieussel-Klötzer-Sava-Huss 2026+]

Under assumptions ($h(Y_n) < \infty$, $\pi(0) = 0$, $\rho > 1$, π second moment)

$$\lim_{n\to\infty} \frac{H(\widehat{M}_n)}{n} = \begin{cases} \log\rho, & \rho \le e^{h(Y_n)}, \\ h(Y_n), & \rho > e^{h(Y_n)}, \end{cases} \qquad \text{almost surely.}$$

Equivalently $\lim_{n\to\infty} \frac{H(\widehat{M}_n)}{n} = \min\{\log\rho, h(Y_n)\}$.

The critical value is

$$\rho_* = e^{h(Y_n)}.$$

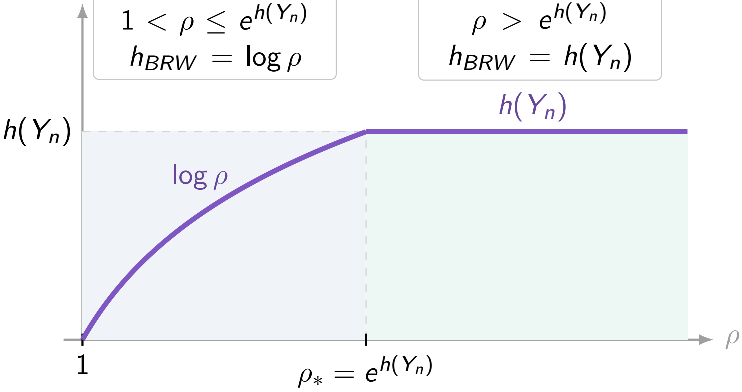
Below ρ_*, population is too small to probe the random-walk distribution; above ρ_*, it is large enough to recover the random-walk entropy rate.

Picture of the phase transition

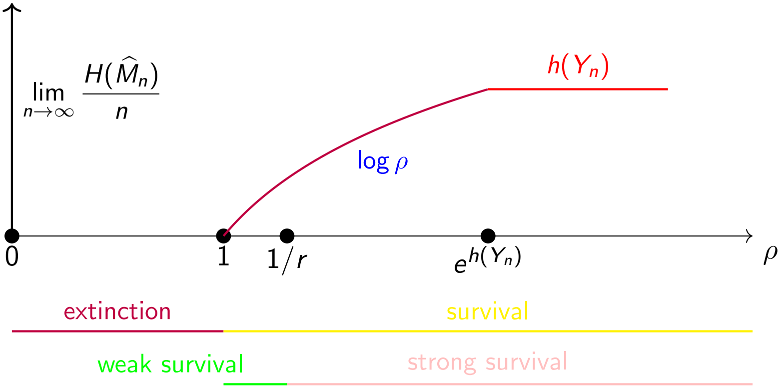
$$h_{BRW} = \lim_{n\to\infty} H(\widehat{M}_n)/n$$

$$1 < \rho \le e^{h(Y_n)} \qquad h_{BRW} = \log \rho$$

$$\rho > e^{h(Y_n)} \qquad h_{BRW} = h(Y_n)$$

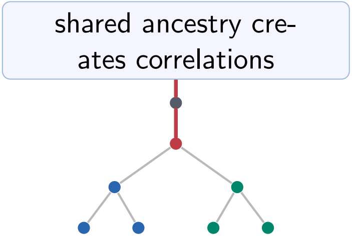


3 phase transitions in branching random walks



BRW is harder than independent random walkers

- BRW particles share long ancestral paths.
- Atypical early particles can create many descendants.
- Standard subadditive tools do not directly apply to $H(\widehat{M}_n)$.
- One must control whole lineages, not only endpoints.



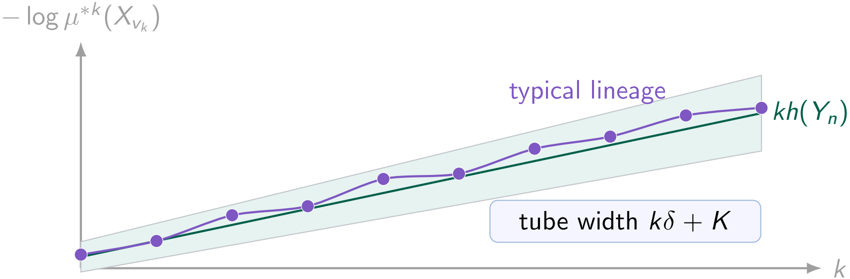
Key proof idea: the entropic tube

For a typical random-walk path, (μ is the step distribution of (Y_n))

$$-\log \mu^{*k}(Y_k) \approx kh(Y_n).$$

Restrict attention to particles whose whole ancestral trajectory stays in this typical window:

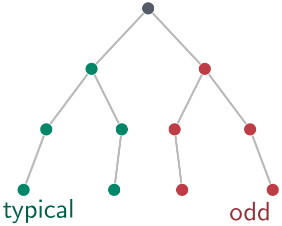
$$\mathrm{Ent}_{n,\delta,K} := \left\{ \forall k \le n : \left| -\log \mu^{*k}(Y_k) - kh(Y_n) \right| \le k\delta + K \right\}.$$



Typical and odd particles

Partition of generation n

- Typ_n: particles whose ancestral paths stay inside the entropic tube.
- Odd_n: particles whose ancestral paths exit the tube.



We prove

$$\frac{\#\mathsf{Typ}_n}{\rho^n} \to W^{\mathrm{typ}}, \quad \frac{\#\mathsf{Odd}_n}{\rho^n} \to W^{\mathrm{odd}}.$$

As the tube is widened, $W^{\mathrm{typ}} \to W$ and $W^{\mathrm{odd}} \to 0$.

Good sites and collision control

The proof then studies how many typical particles may coalesce at the same vertex

Small-ρ regime

When $\rho \leq e^{h(Y_n)}$, typical particles do not cluster too much:

$$M_n^{\mathrm{typ}}(x) \leq (1+c)^n$$

for most of the typical mass. This forces entropy rate $\log \rho$.

Large-ρ regime

When $\rho > e^{h(Y_n)}$, for most typical mass,

$$M_n^{\mathrm{typ}}(x) \lesssim \rho^n e^{-n(h(\mu)-\varepsilon)}.$$

The empirical profile approximates the random-walk entropy scale.

The estimates rely on second-moment bounds for typical-particle overlaps, using the branching time of pairs.

Questions

- Second order estimates for the asymptotic entropy of the empirical distributions of BRW?
- Why is entropy important for understanding the non-trivial behaviour of branching random walks at infinity?

THANK YOU!