On the global dimension of Nakayama algebras

joint work with
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summary

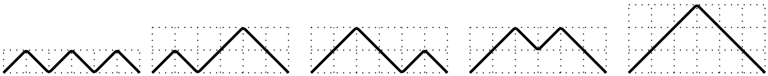
1. open problem
2. proof of the main result
3. combinatorial lemma
4. main result
5. introduction

open problem

$\mathcal{D}_n$ Dyck paths of semilength n

$h(D)$ max. height of Dyck path D

Example



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$g(T)$ max. distance of successive nodes in natural labelling of T

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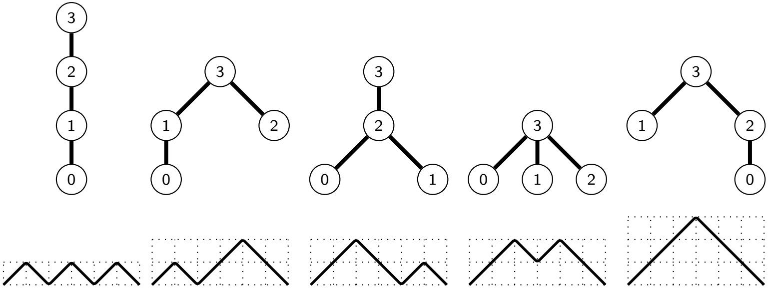

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Problem

Find an explicit bijection $\delta : \mathcal{T}_n \to \mathcal{D}_n$ such that $h \circ \delta = g$.

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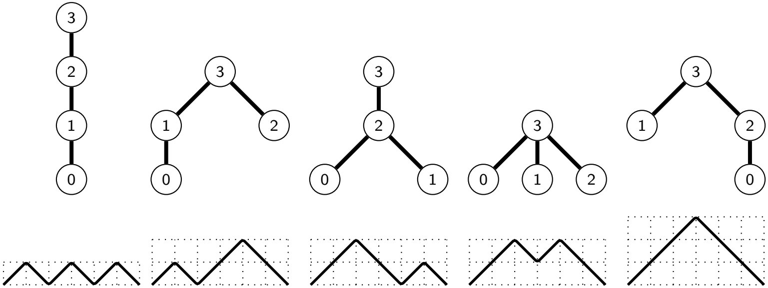


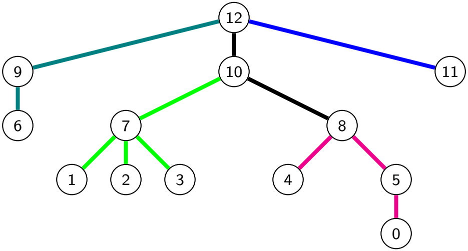
proof of the main result

Theorem

There are explicit bijections

$$\delta_m : \{T \in \mathcal{T}_n \mid g(T) \leq m\} \to \{D \in \mathcal{D}_n \mid h(D) \leq m\}.$$

$m = 5$



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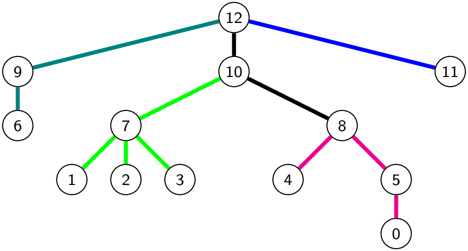

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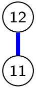

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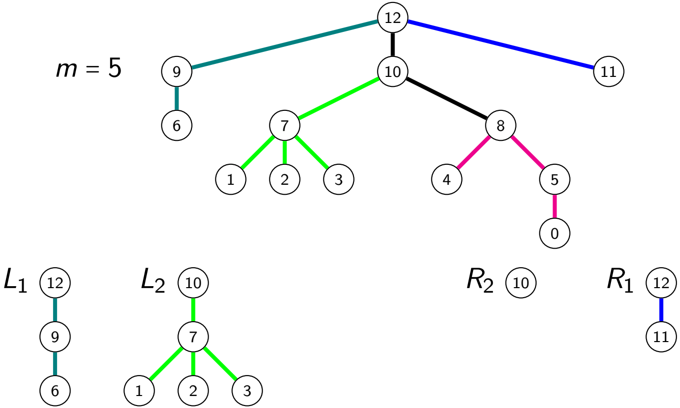

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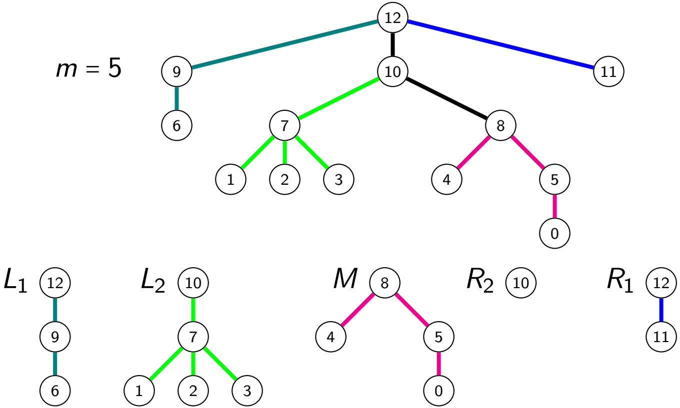

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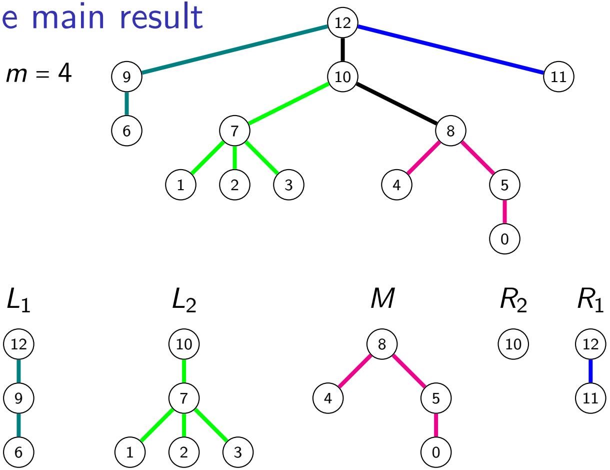

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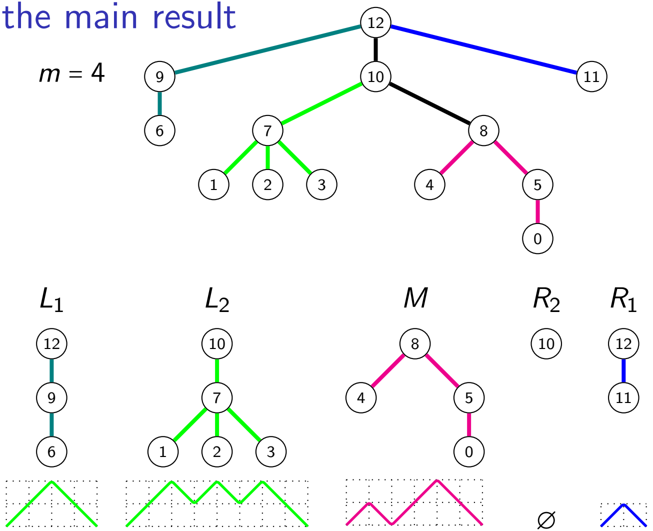

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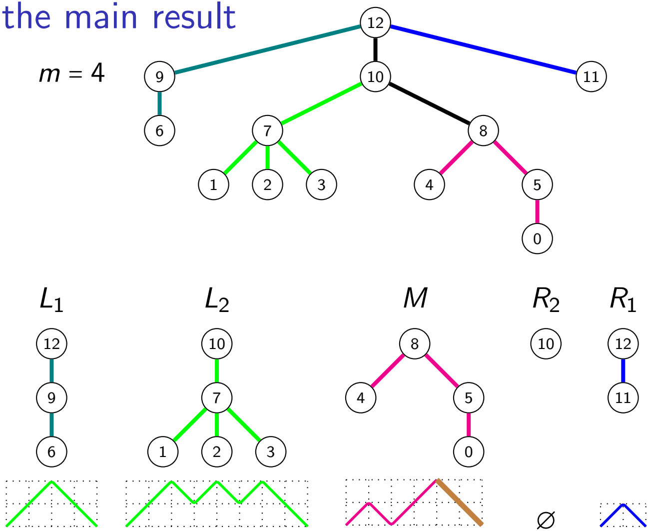

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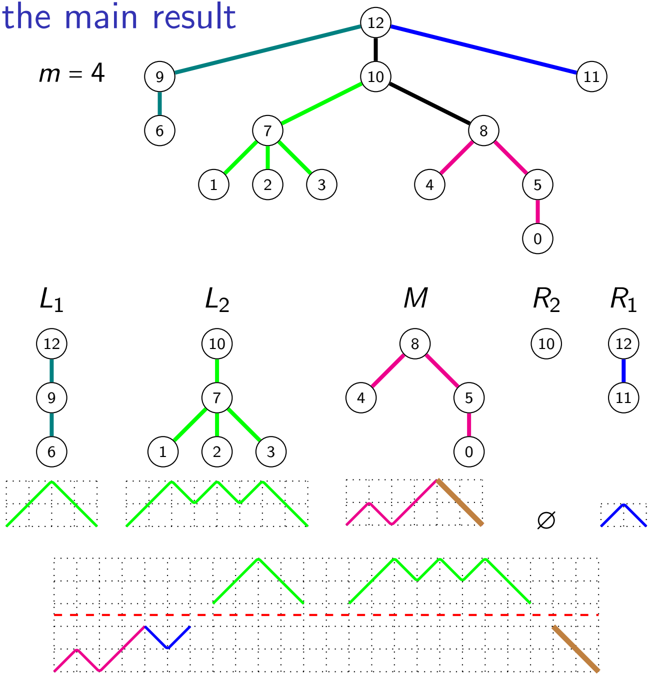
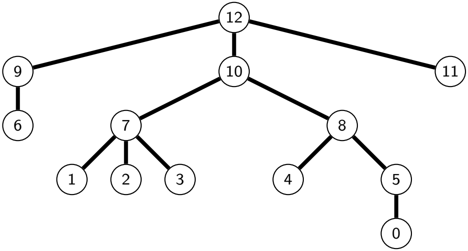


a combinatorial lemma for linear Nakayama algebras

$c_i = \text{parent}(i) - i \quad (0 \leq i < n)$



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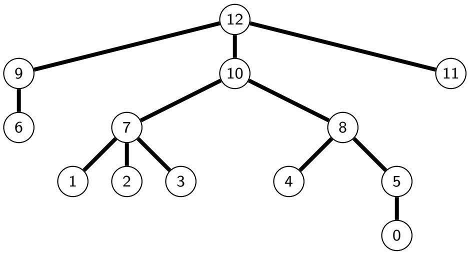

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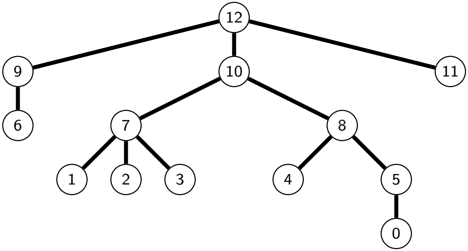

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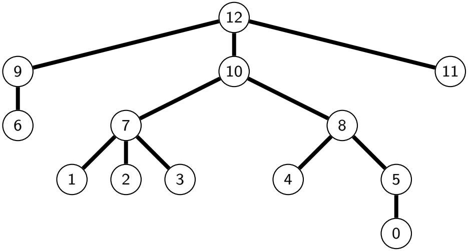

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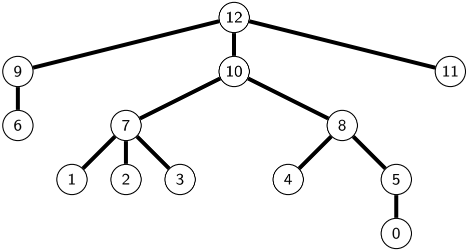

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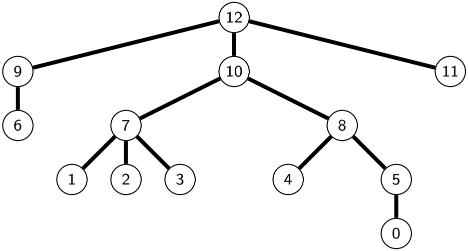

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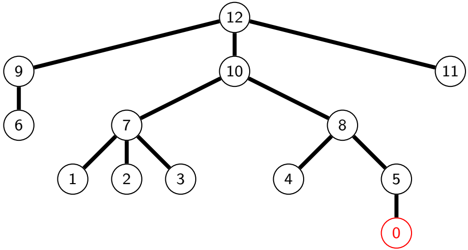

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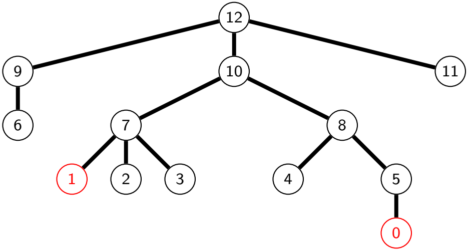

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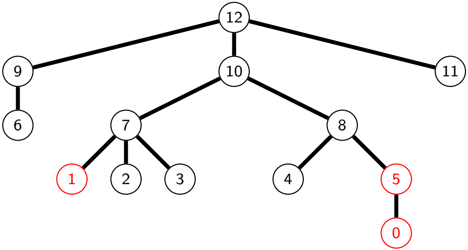

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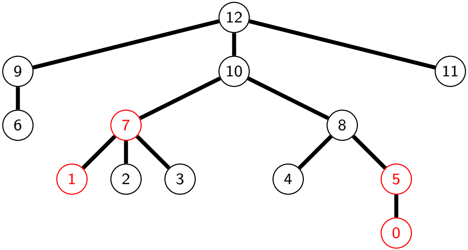

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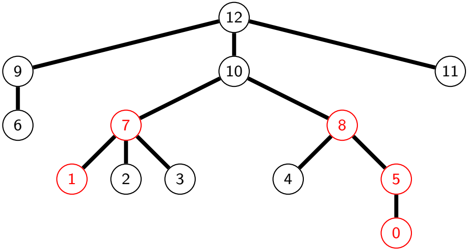

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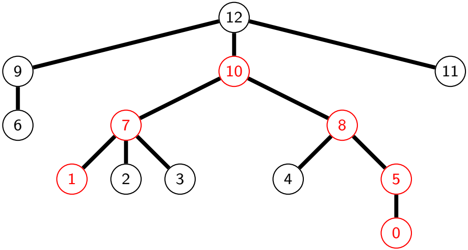

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main result

Theorem

The global dimension of connected linear Nakayama algebras with n simple modules is equidistributed with the height of Dyck paths of semilength $n-1$.

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Remark

In a finite dimensional algebra, the global dimension equals the maximal projective dimension of a simple module.

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Remark

In a finite dimensional algebra, the global dimension equals the maximal projective dimension of a simple module.

Remark

A linear Nakayama algebra with n simple modules is determined up to isomorphism by its Kupisch series $[c_0, \ldots, c_{n-1}]$, satisfying $c_{i+1} + 1 \geq c_i \geq 1$ and $c_{n-1} = 1$. It is connected if only $c_{n-1} = 1$.

introduction

Q linear quiver (= digraph) $\quad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1$

$\mathbb{C}Q$ path algebra, e_i trivial path at vertex i

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$A = \mathbb{C}Q/I$ with $I \subseteq J^2$ is a connected linear Nakayama algebra

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$$Q \text{ cyclic quiver } \quad \begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[llll, bend right] \end{tikzcd}$$

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A connected cyclic Nakayama algebra with n simple modules is determined up to isomorphism by its Kupisch series $[c_0, \ldots, c_{n-1}]$, satisfying $c_{i+1} + 1 \geq c_i \geq 2$, indices taken modulo n.

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which cyclic Nakayama algebras have finite global dim.?
E.g., $n = 3$*:*

$$[2,2,3], [2,4,3], [3,4,4], [3,5,4]$$

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Recently solved by Sven Schmitz (Bonn).