The q, t-symmetry of $(\mathrm{area}, \mathrm{depth})$ for $\vec{k}$-Dyck paths

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Collaborator

This is joint work with...

Figure: Yingrui Zhang, Yunnan University of Finance and Economics.

Contents

$\vec{k}$-Dyck paths

- Let $\vec{k} = (k_1, k_2, \cdots, k_r)$ be a vector of positive integers. We define its length as $\ell(\vec{k}) := r$ and its size as $|\vec{k}| := k_1 + k_2 + \cdots + k_r$.
- Example: $\ell((4, 2, 3, 1)) = 4$ and $|(4, 2, 3, 1)| = 10$.

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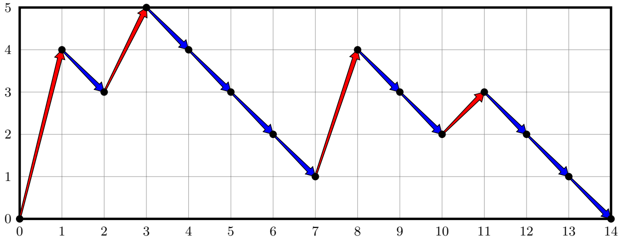
- Example: $\ell((4,2,3,1)) = 4$ and $|(4,2,3,1)| = 10$.
- A $\vec{k}$-Dyck path is a lattice path from $(0,0)$ to $(|\vec{k}| + \ell(\vec{k}), 0)$ that never goes below the horizontal axis, consisting of up steps $U^{k_i} = (1, k_i)$, $1 \leq i \leq \ell(\vec{k})$ appearing in order from left to right, and down steps $D = (1, -1)$. Let $\mathcal{D}_{\vec{k}}$ denote the set of all $\vec{k}$-Dyck paths.
- Example: $\pi = U^4 D U^2 DDDDU^3 DDU^1 DDD \in \mathcal{D}_{(4,2,3,1)}$.



$\mathcal{K}$-Dyck paths

- Let $\mathcal{K}$ be the set of all (multiset) permutations of $\vec{k}$, and let $\mathcal{D}_{\mathcal{K}}$ denote the union of all corresponding path sets $\mathcal{D}_{\vec{k}'}$ over all permutations $\vec{k}'$ in $\mathcal{K}$.
- Example: For $\vec{k} = (2, 2, 1)$, we have

$$\mathcal{K} = \{(2,2,1),(2,1,2),(1,2,2)\}$$
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- These paths generalize:
  - classical Dyck paths when $\vec{k} = (1, 1, \dots, 1)$, where $|\mathcal{D}_{(1^n)}| = \frac{1}{n+1}\binom{2n}{n}$;
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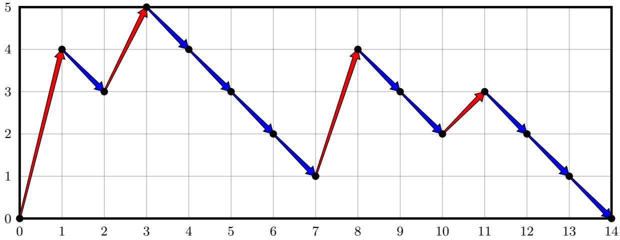
- Rukavicka '11: For $\lambda(\vec{k}) = 1^{m_1}2^{m_2}\cdots|\vec{k}|^{m_{|\vec{k}|}}$, then

$$|\mathcal{D}_{\mathcal{K}}| = \frac{1}{|\vec{k}|+1}\binom{|\vec{k}| + m_1 + \cdots + m_{|\vec{k}|}}{|\vec{k}|, m_1, \cdots, m_{|\vec{k}|}}.$$

area statistic

- For each step π_i, we define its starting rank $r(\pi_i)$ and ending rank $r'(\pi_i)$ as the y-coordinates of its initial and terminal points, respectively.
- Example:

$$\text{starting rank sequence:} r(\pi) = (0, 4, 3, 5, 4, 3, 2, 1, 4, 3, 2, 3, 2, 1),$$
$$\text{ending rank sequence:} r'(\pi) = (4, 3, 5, 4, 3, 2, 1, 4, 3, 2, 3, 2, 1, 0).$$



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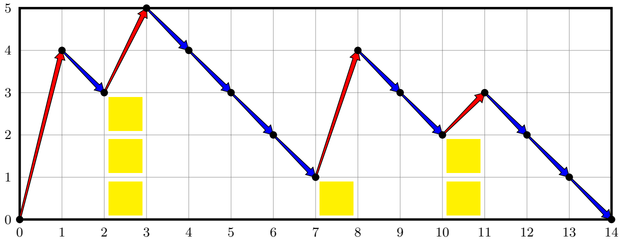

- The area of $\pi \in \mathcal{D}_{\vec{k}}$ is the sum of starting ranks of all up steps. It counts the number of complete cells between the up steps and the horizontal axis.

dinv statistic

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$$\begin{aligned}\mathrm{dinv}(\pi) = &\sum_{D_i < U_j} \chi(0 \le r(D_i) - r(U_j) \le k_j) \\ &+ \sum_{U_i < U_j} \chi(r(U_i) \ge r(U_j) \text{ and } r'(U_j) > r'(U_i))\,(r'(U_j) - r'(U_i)) \\ &+ \sum_{U_i < U_j} \chi(r(U_i) < r(U_j) \text{ and } r'(U_j) < r'(U_i))\,(r'(U_i) - r'(U_j)).\end{aligned}$$

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- For our previous example, we have $\mathrm{dinv}(\pi) = 10$.
  - $(D_1, U_2), (D_1, U_3), (D_3, U_3), (D_4, U_3), (D_4, U_4), (D_5, U_3), (D_5, U_4), (D_7, U_4)$,
  - (U_1, U_4), (U_3, U_4).

bounce statistic

- For $\pi \in \mathcal{D}_{\vec{k}}$, $\mathrm{bounce}(\pi)$ is the sum of the first-row entries of $\gamma(\eta(\pi))$, with η and γ being the filling and ranking algorithms introduced by Xin and Zhang.
- Both $\eta(\pi)$ and $\gamma(\eta(\pi))$ take the shape of a top-justified array of cells consisting of $\ell(\vec{k})$ columns, where the i-th column contains $k_i + 1$ cells.
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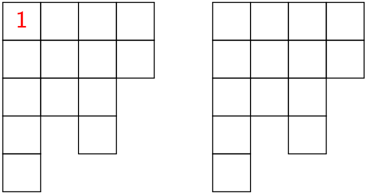

Figure: Filling tableau $\eta(\pi)$ and ranking tableau $\gamma(\eta(\pi))$.

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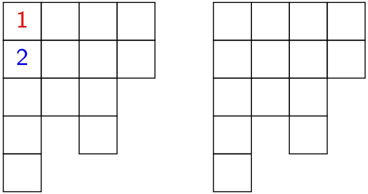

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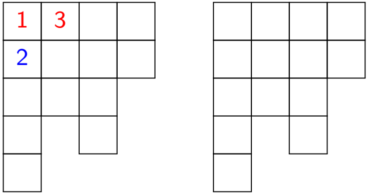

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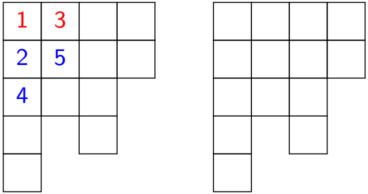

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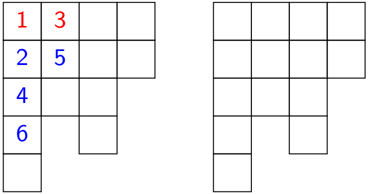

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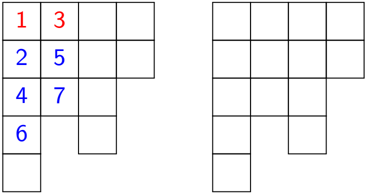

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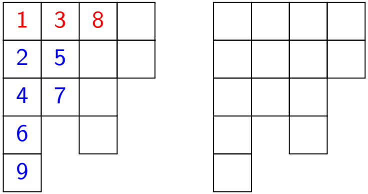

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Thus, $\mathrm{bounce}(\pi) = 0 + 1 + 3 + 4 = 8$.

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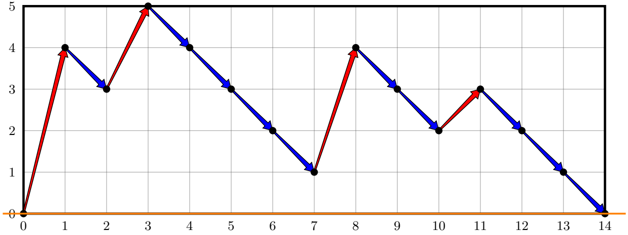

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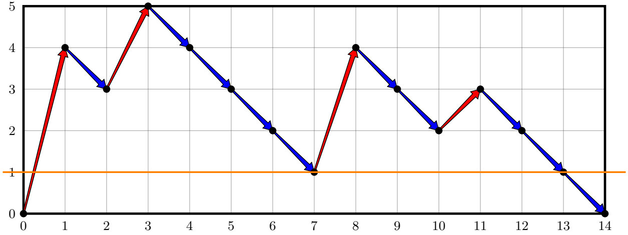

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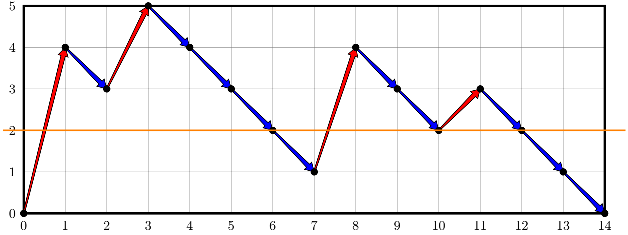

$\zeta(\pi) = (U^4, D, U^3, D, U^1, D, \qquad\qquad\qquad)$

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- Sorting steps according to the starting rank sequence from smallest to largest. For steps starting at the same rank, we list them from right to left.
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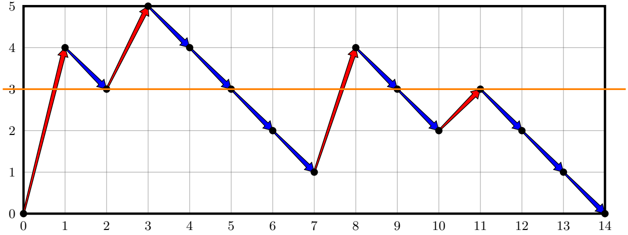

$$\zeta(\pi) = (U^4, D, U^3, D, U^1, D, D, D, D, U^2, \qquad\qquad)$$

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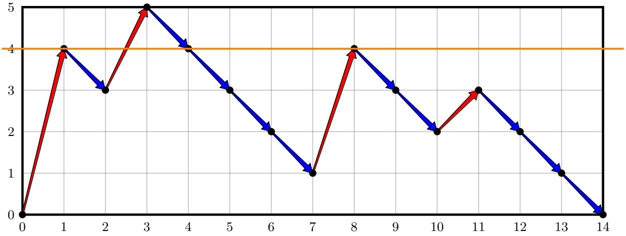

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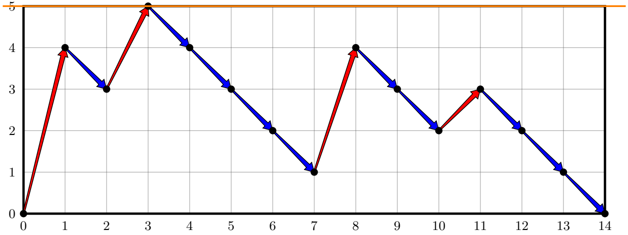

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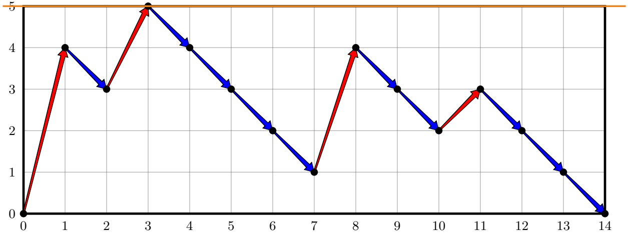

$$\zeta(\pi) = (U^4, D, U^3, D, U^1, D, D, D, D, U^2, D, D, D, D) \in \mathcal{D}_{(4,3,1,2)}.$$

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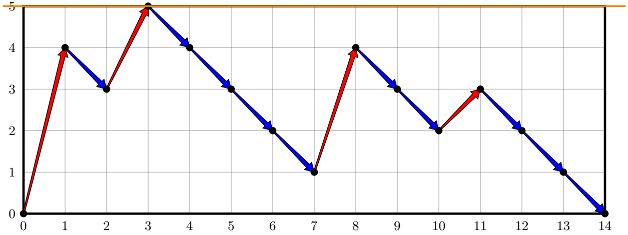

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- Xin–Zhang '21: Sweep map ζ is a bijection from $\mathcal{D}_{\mathcal{K}}$ to $\mathcal{D}_{\mathcal{K}}$ and takes dinv to area and area to bounce, that is,

$$\mathrm{dinv}(\pi) = \mathrm{area}(\zeta(\pi)) \text{ and } \mathrm{area}(\pi) = \mathrm{bounce}(\zeta(\pi)).$$

Symmetry

- Joint symmetry:

$$\sum_{\pi \in \mathcal{D}_{\mathcal{K}}} q^{\mathrm{dinv}(\pi)} t^{\mathrm{area}(\pi)} = \sum_{\pi \in \mathcal{D}_{\mathcal{K}}} q^{\mathrm{area}(\pi)} t^{\mathrm{bounce}(\pi)}.$$

- The dinv, area, and bounce have the same distribution on $\mathcal{D}_{\mathcal{K}}$ for any $\vec{k}$.

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  - Garsia–Haglund '02: $C_{(1^n)}(q,t) = C_{(1^n)}(t,q) = \langle \nabla e_n, e_n \rangle$.
  - Loehr '05: $C_{(k^n)}(q,t) = C_{(k^n)}(t,q) = \langle \nabla^k e_n, e_n \rangle$.

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  - Xin–Zhang '24: $C_{\mathcal{K}}(q,t) = C_{\mathcal{K}}(t,q)$ for $\ell(\vec{k}) \leq 3$.
  - It does NOT hold in general for $\ell(\vec{k}) \geq 4$. For example, $\vec{k} = (3,1,1,1)$:

$$\begin{aligned} C_{\mathcal{K}}(q,t) - C_{\mathcal{K}}(t,q) = {} & q^6t^3 - q^3t^6 - q^6t^2 - 2q^5t^3 + 2q^3t^5 + q^2t^6 \\ & + 2q^5t^2 + q^4t^3 - q^3t^4 - 2q^2t^5 - q^4t^2 + q^2t^4. \end{aligned}$$

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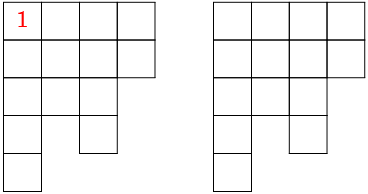

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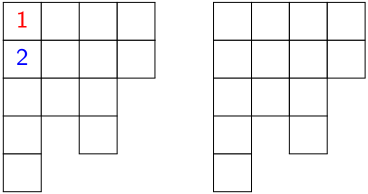

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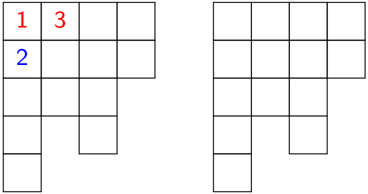

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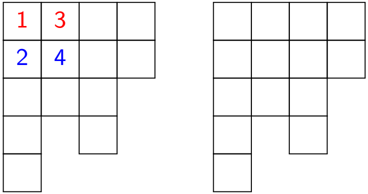

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1	3		
2	4		
	5		

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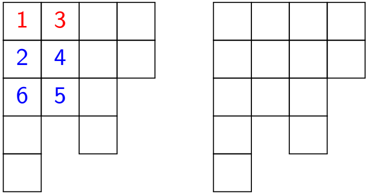

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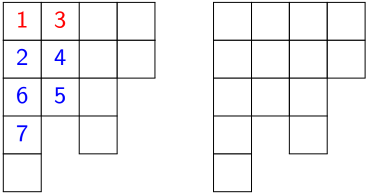

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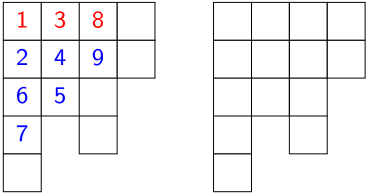

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1	3	8	11
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0	1		
1	2		
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2	3	5	
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Thus, $\mathrm{depth}(\pi) = 0 + 1 + 3 + 5 = 9$.

depth statistic

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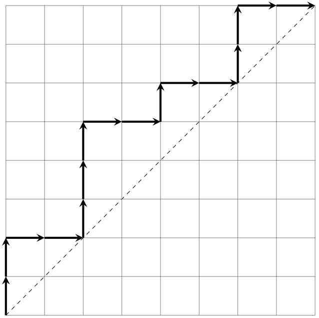

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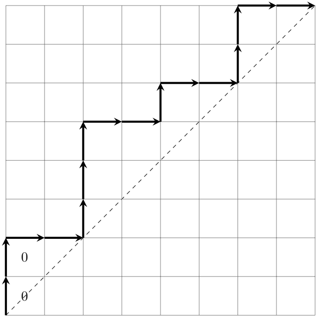

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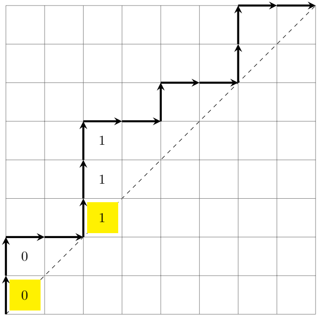

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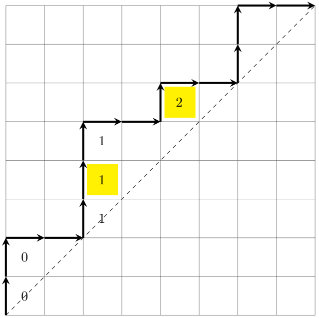

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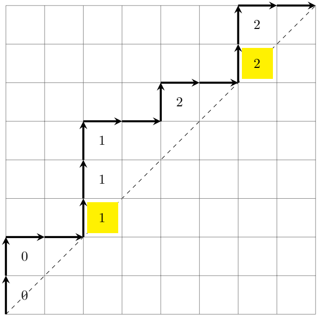

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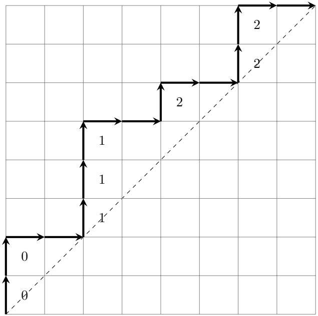
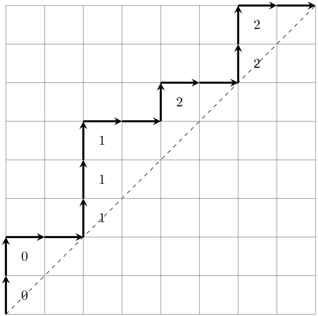


depth statistic

- Pappe–Paul–Schilling's depth for classical Dyck paths:



- Example: For $\pi = U^1U^1DDU^1U^1U^1DDU^1DDU^1U^1DD \in \mathcal{D}_{(1^8)}$.

1	2	5	6	7	10	13	14
4	3	12	9	8	11	16	15

0	0	1	1	1	2	2	2
1	1	2	2	2	3	3	3

Figure: Modified filling tableau $\eta_*(\pi)$ and ranking tableau $\gamma_*(\eta_*(\pi))$.

(area,depth)-polynomials

- Define
$$\widetilde{C}_{\vec{k}}(q,t) := \sum_{\pi \in \mathcal{D}_{\vec{k}}} q^{\mathrm{area}(\pi)} t^{\mathrm{depth}(\pi)}.$$

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Conjecture (Q.–Zhang '26)

For any $\vec{k}$ and positive integers a and b, then $\widetilde{C}_{{}^a\mathcal{K}^b}(q,t) = \widetilde{C}_{{}^a\mathcal{K}^b}(t,q)$.

- Example: Let $\vec{k} = (2,2,1)$, $a = 3$ and $b = 1$, then

$${}^3\mathcal{K}^1 = \{(3,2,2,1,1),(3,2,1,2,1),(3,1,2,2,1)\}.$$

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Theorem (Q.–Zhang '26)

For any $\vec{k}$ and positive integer a, then $\widetilde{C}_{{}^a\mathcal{K}}(q,t) = \widetilde{C}_{{}^a\mathcal{K}}(t,q)$.

Corollary (Q.–Zhang '26)

For any $\vec{k}$, then $\widetilde{C}_{\mathcal{K}}(q,t) = \widetilde{C}_{\mathcal{K}}(t,q)$.

Contents

area and depth: recursive definitions

- Let $v_1, v_2, \cdots, v_\ell$ be the first-row entries of the modified filling tableau. Fix v_j, and assume $v_j - 1$ appears in the $m_{i,j}$-th row of the i-th column for some $i < j$. Then, the area sequence $a(\pi) = (a_1, a_2, \ldots, a_\ell)$ satisfies
  - $a_1 = 0$,
  - $a_j = a_i + k_i + 1 - m_{i,j}$ for $j \geq 2$.

Additionally, the depth labeling sequence $d(\pi) = (d_1, d_2, \ldots, d_\ell)$ satisfies

- $d_1 = 0$,
- $d_j = d_i + m_{i,j} - 1$ for $j \geq 2$.

$$
\begin{array}{rcccccc}
 & & i & & j & \\
1 \longrightarrow & \cdots & \boxed{v_i} & \cdots & \boxed{v_j} & \cdots \\
 & & \vdots & & \vdots & \\
m_{i,j} \longrightarrow & \cdots & \boxed{v_j - 1} & \cdots & \cdots & \cdots \\
 & & \vdots & & \vdots & \\
k_i + 1 \longrightarrow & \cdots & \boxed{*} & & \vdots &
\end{array}
$$

1	3	8	11
2	4	9	12
6	5	10	
7		13	
14			

From modified filling tableau to labeled branch tree

- Example: For $\pi = {\color{red}U^4}{\color{blue}DU^2DDDD}{\color{red}U^3}{\color{blue}DDU^1DDD} \in \mathcal{D}_{(4,2,3,1)}$,

1	3	8	11
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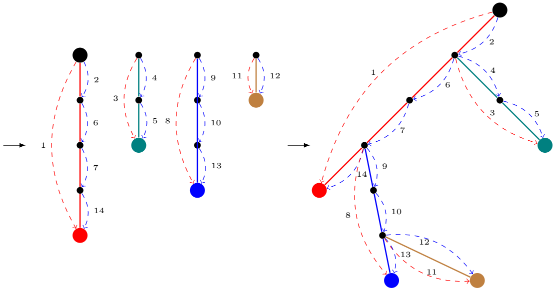

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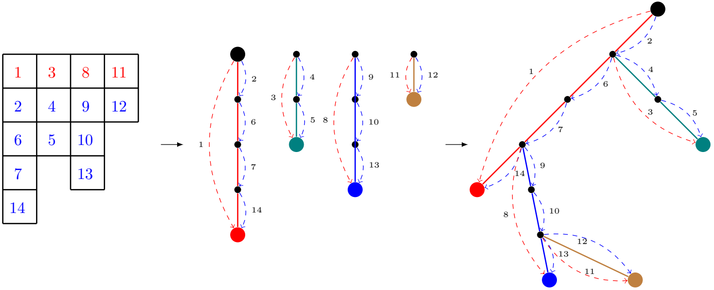
- Example: For $\pi = U^4 D U^2 DDDDU^3 DDU^1 DDD \in \mathcal{D}_{(4,2,3,1)}$,

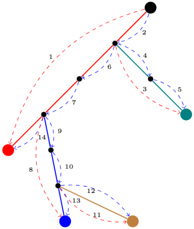


- Define area and depth on labeled branch trees recursively as follows:

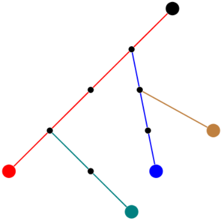
$$a_1(T) := 0 \text{ and } a_j(T) := a_i(T) + e_{i,j},$$
$$d_1(T) := 0 \text{ and } d_j(T) := d_i(T) + s_{i,j}.$$

- We have $\mathrm{area}(T) = \mathrm{area}(\pi)$ and $\mathrm{depth}(T) = \mathrm{depth}(\pi)$.

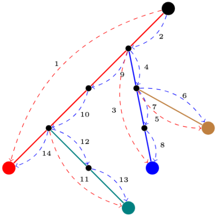
Dual algorithm



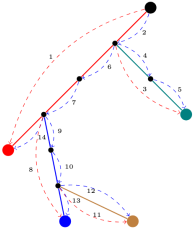
$\Rightarrow$



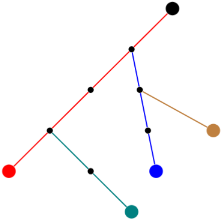
$\Rightarrow$



Dual algorithm



$\Rightarrow$



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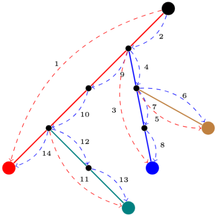
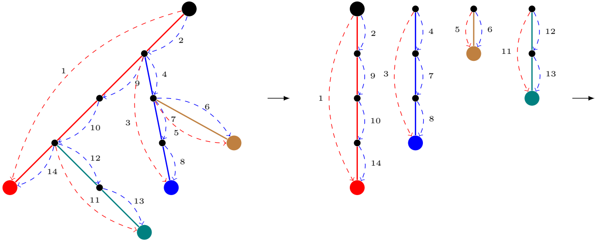


- The dual is an involution. Furthermore, we have

$$\text{area}(T^{\text{dual}}) = \text{depth}(T) \text{ and } \text{depth}(T^{\text{dual}}) = \text{area}(T).$$

From labeled branch tree to modified filling tableau



1	3	5	11
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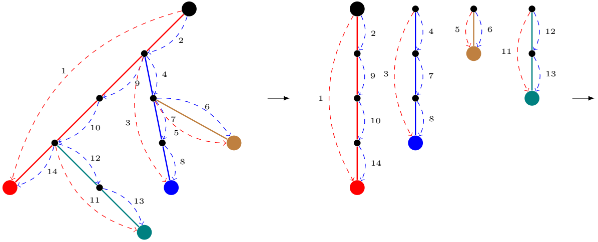
- Let ω be the composition of the mapping steps described above, then

$$\omega(\pi) = U^4 D U^3 D U^1 DDDDD U^2 DDD.$$

From labeled branch tree to modified filling tableau



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- ω defines an involution on $\mathcal{D}_{a\mathcal{K}}$ which interchanges area and depth.

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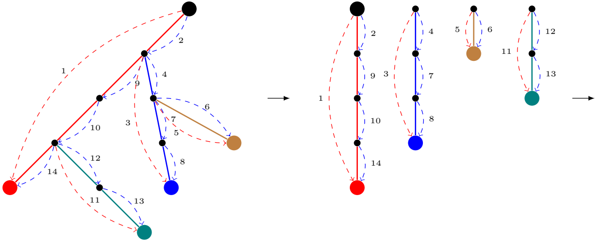

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$$\omega(\pi) = U^4DU^3DU^1DDDDDU^2DDD.$$

- ω defines an involution on $\mathcal{D}_{^a\mathcal{K}}$ which interchanges area and depth.
- The q,t-symmetry of $\widetilde{C}_{\mathcal{K}'}(q,t)$ follows immediately from the fact that it is a sum of q,t-symmetric components $\widetilde{C}_{^a\mathcal{K}}(q,t)$.

Thanks for your time!