On the small-step quarter-plane walks with a non-D-finite univariate generating function

Enumeration, D-finiteness, and boundary asymptotics

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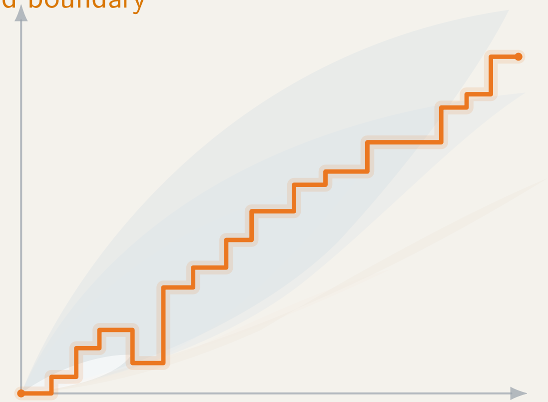
Lattice Path Conference



Based on joint work
with Marni Mishna and Lucas Pannier.

Quarter-plane walks

The model

A *small-step model* is a set

$$S \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}.$$

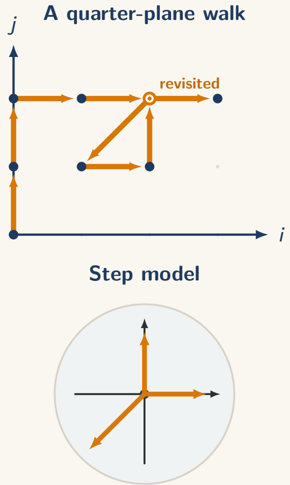
A quarter-plane walk of length n, starting at the origin, is a sequence

$$\omega = (s_1, s_2, \ldots, s_n), \qquad s_k \in S,$$

such that every partial sum remains in the quarter plane:

$$s_1 + \cdots + s_k \in \mathbb{Z}^2_{\geq 0} \qquad (1 \leq k \leq n).$$

The walks may self-intersect.



The endpoint generating function

Endpoint generating function

Let $q(i, j; n)$ denote the number of quarter-plane walks of length n ending at (i, j). We define

$$Q_S(x, y; t) = \sum_{i,j,n \geq 0} q(i, j; n) x^i y^j t^n.$$

Here t records the length, while $x^i y^j$ records the endpoint (i, j).

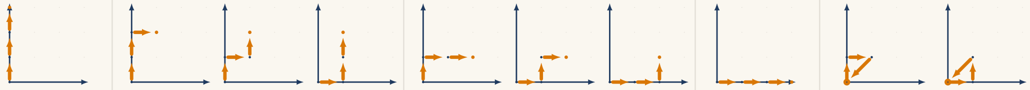
Ends at $(0, 3)$
1 **walk**

Ends at $(1, 2)$
3 **walks**

Ends at $(2, 1)$
3 **walks**

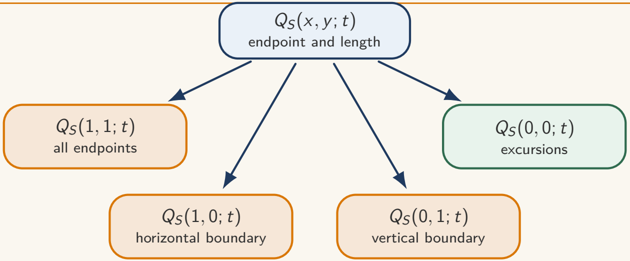
Ends at $(3, 0)$
1 **walk**

Ends at $(0, 0)$
2 **excursions**



$$\boxed{[t^3] Q_S(x, y; t) = y^3 + 3xy^2 + 3x^2y + x^3 + 2}$$

Four natural specializations



Univariate classification problem [Bousquet-Mélou–Mishna, 2010]

For each of the small-step models, determine the nature of these generating functions:

algebraic, D-finite, or non-D-finite?

MAIN FOCUS

$$Q_S(1, 1; t)$$

the generating function counting all quarter-plane walks

The 79 relevant small-step models

3 steps

4 steps

5 steps

6 steps

7–8 steps

After removing trivial cases and identifying models related by the symmetry $x \leftrightarrow y$, exactly 79 models remain.
[Bousquet-Mélou–Mishna, 2010]

Complete classification of the endpoint series

To every small-step model S, one associates its *group of the walk* $G(S)$.

COMPLETE CLASSIFICATION

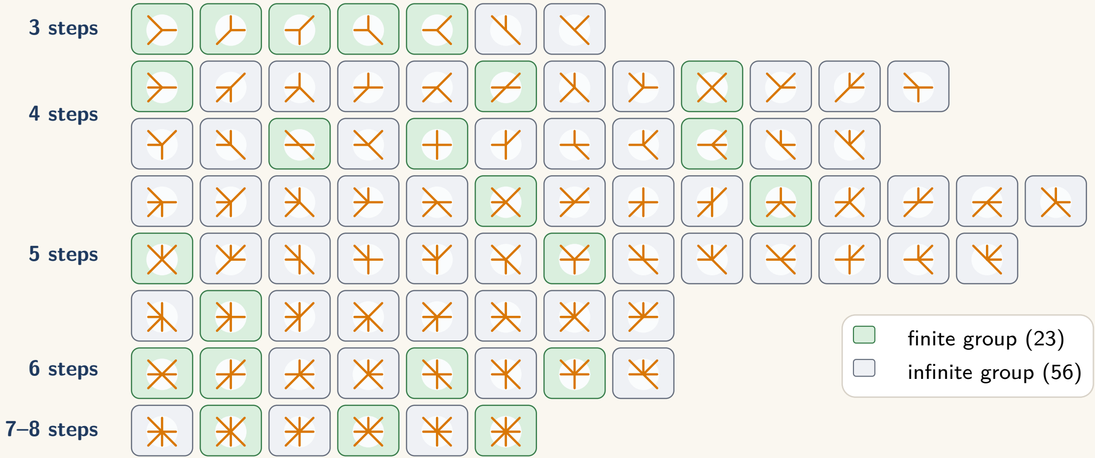
$$Q_S(x, y; t) \text{ is D-finite} \iff G(S) \text{ is finite}$$

23
finite-group models
$Q_S(x, y; t)$ is D-finite

56
infinite-group models
$Q_S(x, y; t)$ is non-D-finite

Finite group: Bousquet-Mélou–Mishna; Bostan–Kauers. Infinite group: Dreyfus–Elvey Price–Raschel; Hardouin.

Endpoint classification of the 79 models



Why the univariate specializations remain open

Non-D-finiteness may vanish after specialization

Let $B(t)$ be non-D-finite and define

$$F(x,y;t) = \frac{1}{1-t} + \underbrace{(xy-1)B(t)}_{\text{non-D-finite contribution}} .$$

Hence $F(x,y;t)$ is non-D-finite. However, at $x=y=1$,

$$\boxed{F(1,1;t) = \frac{1}{1-t}}$$

$Q_S(1,1;t)$	$Q_S(1,0;t)$	$Q_S(0,1;t)$	$Q_S(0,0;t)$
all endpoints	horizontal boundary	vertical boundary	excursions
open in general	**open in general**	**open in general**	**classified**

Excursion classification: Bostan–Raschel–Salvy for nonsingular models; the singular cases have $Q_S(0,0;t)=1$.

Asymptotic obstructions to D-finiteness

The infinite-group case for $Q_S(1, 1; t)$

Two central theorems

Theorem (Bostan–Raschel–Salvy, 2014)

Let

$$F(t) = \sum_{n\geq 0} a_n t^n, \qquad a_n \in \mathbb{Z}_{\geq 0}.$$

If

$$a_n \sim C\rho^n n^{-\alpha}, \qquad C > 0, \qquad \alpha \notin \mathbb{Q},$$

then $F(t)$ is non-D-finite.

Theorem (Bostan–Raschel–Salvy, using Denisov–Wachtel)

For each of the 51 nonsingular infinite-group models,

$$[t^n]Q_S(0,0;t) \sim C_S \rho_S^n n^{-\alpha_{00}},$$

where

$$\boxed{\alpha_{00} = 1 + \frac{\pi}{\theta} \notin \mathbb{Q}}$$

and θ is an angle determined by the model.

From counting to probability

Choose a step sequence uniformly

There are exactly

$$|S|^n$$

sequences of n steps chosen from S.

Let τ denote the first time that the corresponding walk leaves the quarter plane. Then

$$\mathbb{P}(\tau > n) = \frac{\#\{\text{quarter-plane walks of length } n\}}{|S|^n}.$$

Therefore,

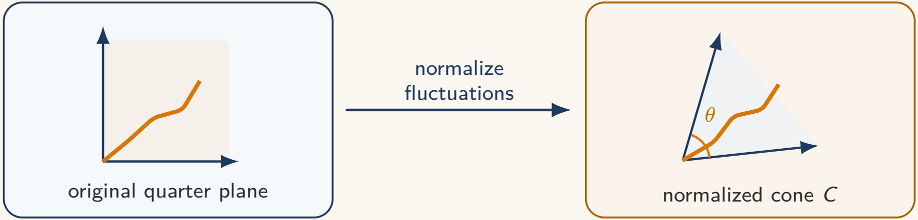
$$\boxed{[t^n]Q(1,1;t) = |S|^n\,\mathbb{P}(\tau > n)}.$$

The dictionary

coefficient asymptotics $\longleftrightarrow$ survival-probability asymptotics.

Why a cone appears

The two coordinates may fluctuate at different scales and may be correlated.



Geometric effect of normalization

Normalizing these fluctuations stretches and shears the plane. The two coordinate axes become two rays, so the quarter plane becomes a cone C with opening angle θ.

The cone angle will provide information of the asymptotic exponents.

Zero drift: Denisov–Wachtel

Theorem (Denisov–Wachtel, 2015)

Let

$$\delta = \frac{1}{|S|} \sum_{s \in S} s$$

be the drift vector of the random walk. If $\delta = (0,0)$*, then for the associated cone of angle* θ*,*

$$\mathbb{P}(\tau > n) \sim \kappa n^{-\beta/2}, \qquad \beta = \frac{\pi}{\theta}.$$

Using $[t^n]Q_S(1,1;t) = |S|^n \mathbb{P}(\tau > n)$*,*

$$[t^n]Q_S(1,1;t) \sim C|S|^n n^{-\alpha_{11}}, \qquad \boxed{\alpha_{11} = \frac{\beta}{2}}.$$

What we verify

For the three zero-drift models:

- $\delta = (0,0)$;
- the walk is genuinely two-dimensional;
- finite range is automatic.

Exponent transfer

For excursions,

$$\alpha_{00} = 1 + \beta.$$

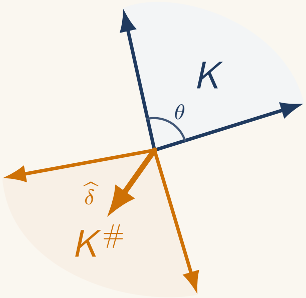
Hence

$$\boxed{\alpha_{11} = \frac{\alpha_{00} - 1}{2} \notin \mathbb{Q}.}$$

Therefore $Q_S(1,1;t)$ is non-D-finite.

Duraj regime: hypotheses and geometry

For the 13 polar-interior models, we identify the models and verify Duraj's hypotheses.



The decisive geometry is $\widehat{\delta} \in \mathrm{int}(K^{\#})$.
K is the original cone.

1. Nonzero polar drift

$$\widehat{\delta} \neq 0, \qquad \widehat{\delta} \in \mathrm{int}(K^{\#}).$$

2. Cramér centering
Let X denote one random step chosen from S.

$$R(h) = \mathbb{E}\left(e^{\langle h, X\rangle}\right), \quad \exists\, h_0 \in (0,\infty)^2 : \nabla R(h_0) = 0.$$

After tilting by h_0, the walk has drift zero.

3. Standard walk assumptions
finite exponential moments; genuine two-dimensionality; strong aperiodicity; regular cone.

Duraj regime: consequence

Theorem (Duraj, 2014; two-dimensional case)

Under the hypotheses from the previous slide,

$$\mathbb{P}(\tau > n) \sim \kappa \gamma^n n^{-(1+\beta)}, \qquad \beta = \frac{\pi}{\theta}.$$

Survival probability

$$\mathbb{P}(\tau > n) \sim \kappa \gamma^n n^{-(1+\beta)}$$

$\longrightarrow$

Enumeration

$$[t^n] Q_S(1,1;t) = |S|^n \mathbb{P}(\tau > n) \sim C(|S|\gamma)^n n^{-\alpha_{11}}$$

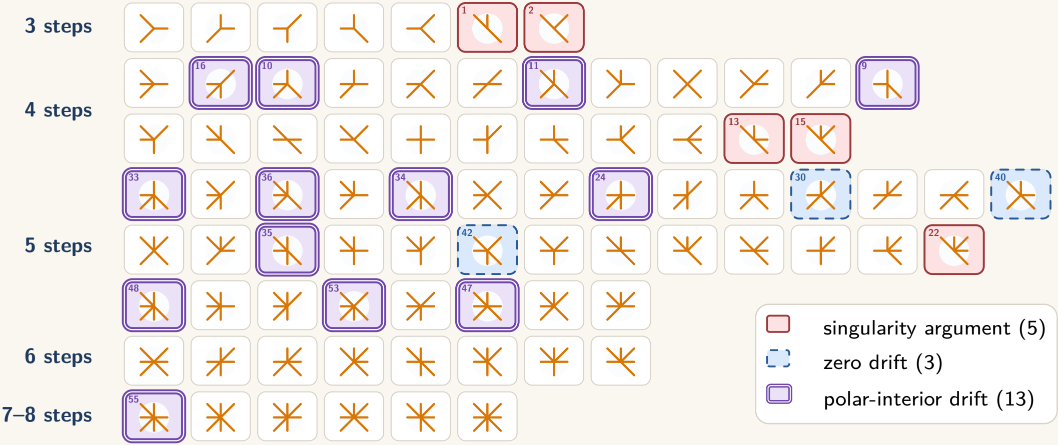
The same exponent as excursions

Since the excursion exponent is $\alpha_{00} = 1 + \beta$, Duraj gives

$$\boxed{\alpha_{11} = 1 + \beta = \alpha_{00} \notin \mathbb{Q}.}$$

Therefore $Q_S(1,1;t)$ is non-D-finite for the 13 polar-interior models.

The 21 proved cases for $Q_S(1,1;t)$



For every highlighted model, $Q_S(1,1;t)$ is non-D-finite.

Singularities: Mishna–Rechnitzer; Melczer–Mishna. Cone asymptotics: Denisov–Wachtel; Duraj.

From proved cases to the remaining models

Boundary asymptotics as the next strategy

The first 21 infinite-group cases are settled.
What can be said about the rest?

Functional equation: reduction to the boundary

Kernel notation

$$P_S(x,y) = \sum_{(i,j)\in S} x^i y^j, \qquad K(x,y;t) = xy\bigl(1 - tP_S(x,y)\bigr).$$

Functional equation

$$K(x,y;t)\,Q_S(x,y;t) = xy + K(x,0;t)\,Q_S(x,0;t) + K(0,y;t)\,Q_S(0,y;t) - K(0,0;t)\,Q_S(0,0;t)$$

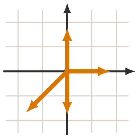
Specializing to $x = y = 1$

$$\boxed{(1-|S|t)\,Q_S(1,1;t) = 1 + K(1,0;t)\,Q_S(1,0;t) + K(0,1;t)\,Q_S(0,1;t) - K(0,0;t)\,Q_S(0,0;t)}$$

The total series is linked to the two boundary series and the excursion series.

A representative numerical pattern: Model 3

Model 3



For each specialization $F(t)$, we estimate

$$[t^n]F(t) \sim C\rho^n n^{-\alpha}.$$

Estimated singular exponents

near $3/2$	$Q_S(1,1;t)$ 1.5051	$Q_S(1,0;t)$ 1.5172
cone-type pair	$Q_S(0,1;t)$ 2.6086	$Q_S(0,0;t)$ 2.6106

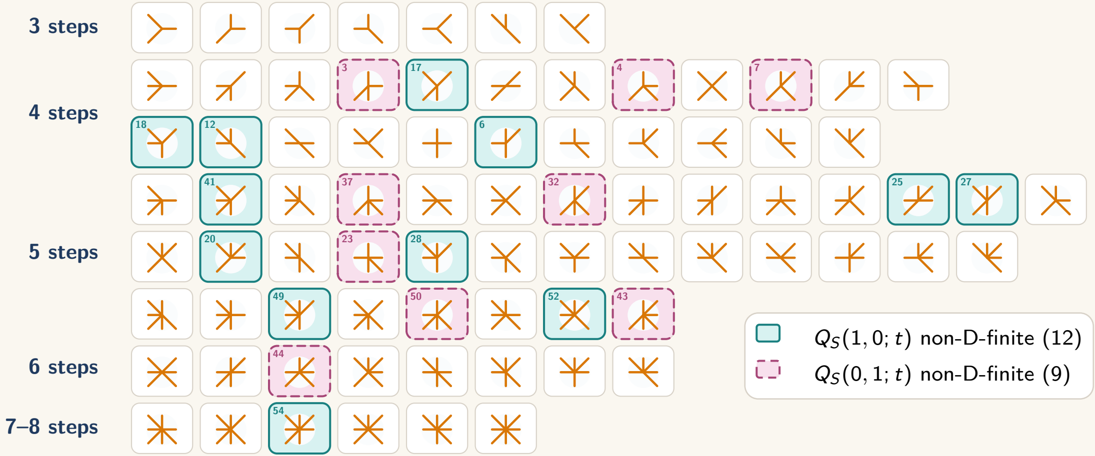
Numerical reading

The excursion exponent α_{00} is known to be irrational. This points to

$$\boxed{Q_S(0,1;t)}$$

as the boundary series carrying the non-D-finite obstruction.

The 21 boundary-series cases



Central weightings

Endpoint-only weights

Give each step (i,j) weight $a^i b^j$, $a, b > 0$. If a walk ends at (r,s), then

$$\mathrm{wt}_{a,b}(\omega) = a^r b^s.$$

Equivalently,

$$\boxed{Q_S^{(a,b)}(x,y;t) = Q_S(ax,by;t).}$$

Vertical weighting
Set $a = 1$. Then

$$Q_S^{(1,b)}(1,0;t) = Q_S(1,0;t).$$

Horizontal weighting
Set $b = 1$. Then

$$Q_S^{(a,1)}(0,1;t) = Q_S(0,1;t).$$

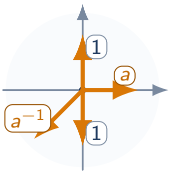
Strategy

Change the drift while preserving the boundary series suggested by the numerical data.

Model 3: the admissible weighting interval

Horizontal weighting

$$w_a(i,j) = a^i.$$



The unnormalized drift is

$$\widetilde{\delta}(a) = \left(a - a^{-1}, -a^{-1}\right).$$

Since $\mathrm{int}(K^{\#}) = (-\infty, 0)^2$, the polar-interior condition becomes

$$\widetilde{\delta}(a) \in \mathrm{int}(K^{\#}) \iff \boxed{0 < a < 1}.$$

Cramér condition

Solving the critical-point equations gives

$$(x_a, y_a) = \left(\frac{z_0}{a}, \frac{1}{z_0^2}\right),$$

where $z_0 \approx 0.81917$. The Cramér vector

$$h_a = (\log x_a, \log y_a)$$

must lie in $(0,\infty)^2$. Since $z_0 < 1$, $y_a = \frac{1}{z_0^2} > 1$, while

$$x_a = \frac{z_0}{a} > 1 \iff a < z_0.$$

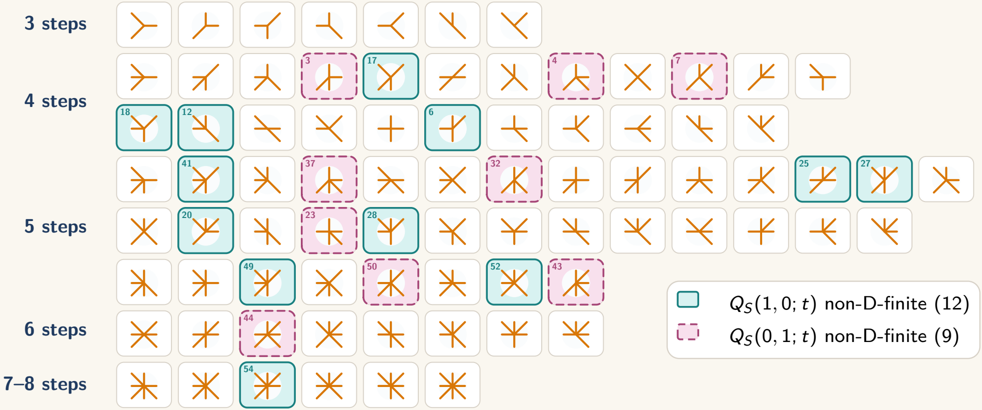
Hence

$$\boxed{0 < a < z_0}.$$

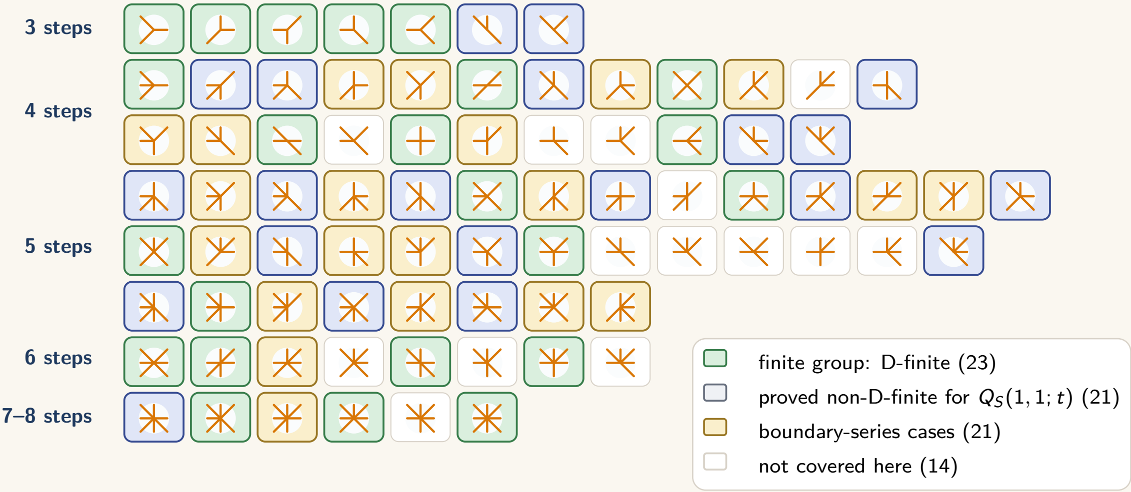
Boundary non-D-finiteness for 21 models

Theorem (Mishna–P.–Pannier)

For every highlighted infinite-group model, the indicated boundary series is non-D-finite.



Where the classification stands



What to remember

21

proved for

$Q_S(1,1;t)$

singularities and cone asymptotics

21

boundary obstructions

directional weightings preserve $Q(1,0)$ or $Q(0,1)$

14

remaining

new ideas are still needed

Next stages

Transfer the 21 boundary obstructions to the total series

$$Q_S(1,1;t),$$

and develop new ideas for the remaining 14 models.

Thank you!

Scan for the article.