# Combinatorics of nondeterministic walks

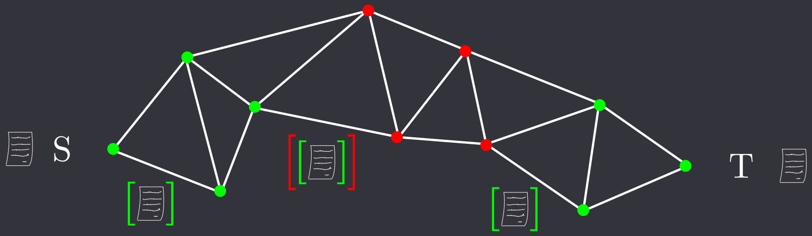
Élie de Panafieu, Lamine Lamali, Michael Wallner

Nokia Bell Labs, Labri Networks team, TU Wien

Lattice Path Conference 2026

# Motivation: networks and encapsulations

Encapsulation of the message to go through parts of the network.




Probability of transmissibility for random encapsulation abilities?

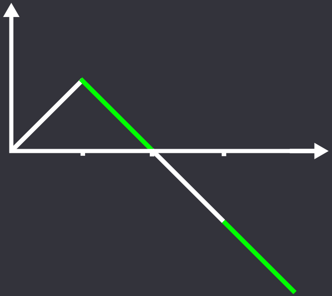
Simplified model: line graph, only one protocole

- explore what is achievable,
- find relevant mathematical tools.

# Non-deterministic walks

Sequence of sets of steps, called N-steps.

N-walk $\{1\}\{-1, 1\}\{-1\}\{-1, 1\}$

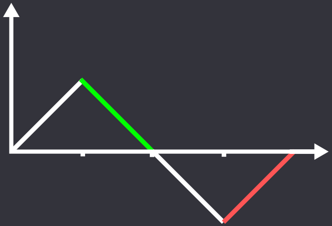


Counting N-walks that contain a walk satisfying a constraint.

# Non-deterministic walks

Sequence of sets of steps, called N-steps.

N-walk $\{1\}\{-1, 1\}\{-1\}\{-1, 1\}$

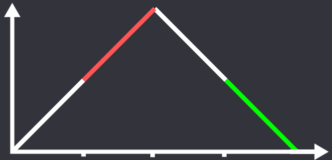


Counting N-walks that contain a walk satisfying a constraint.

# Non-deterministic walks

Sequence of sets of steps, called N-steps.

N-walk $\{1\}\{-1,1\}\{-1\}\{-1,1\}$

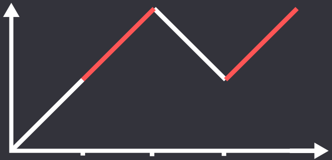


Counting N-walks that contain a walk satisfying a constraint.

# Non-deterministic walks

Sequence of sets of steps, called N-steps.

N-walk $\{1\}\{-1,1\}\{-1\}\{-1,1\}$

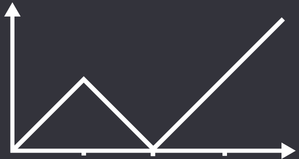


Counting N-walks that contain a walk satisfying a constraint.

# Non-deterministic Dyck walks

## Dyck

steps {-1,1}



walk 1 -1 1 1

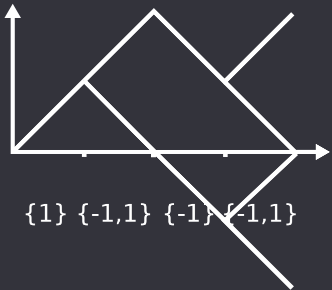
bridge: ends at 0

meander: stays nonnegative

excursion: ends at 0 and stays nonnegative

## N-Dyck

N-steps {{-1},{1},{-1,1}}



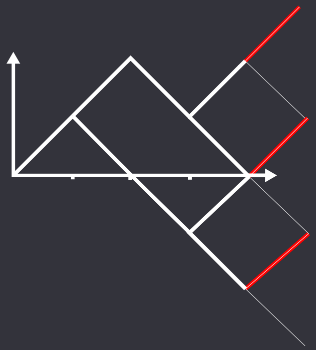
N-walk {1} {-1,1} {-1} {-1,1}

N-bridge: contains a bridge

N-excursion: contains an excursion

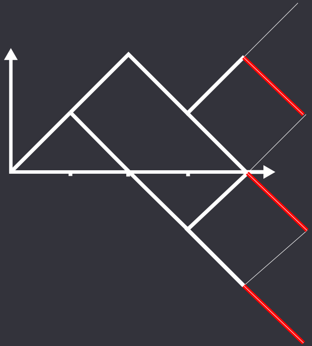
Reachable points: {-2, 0, 2}

# Dyck N-bridges



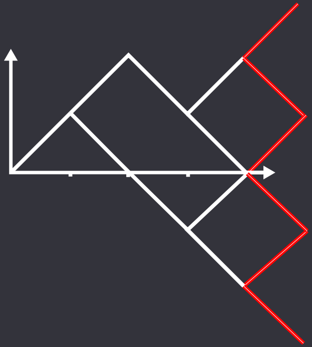
- Between the minimum and maximum reachable points, one over two points is reachable,
- N-bridges have even length.

# Dyck N-bridges



- Between the minimum and maximum reachable points, one over two points is reachable,
- N-bridges have even length.

# Dyck N-bridges



- Between the minimum and maximum reachable points, one over two points is reachable,
- N-bridges have even length.

# Dyck N-bridges

Generating function of Dyck N-walks

$$D(x, y; t) = \sum_{w} x^{\min(w)} y^{\max(w)} t^{|w|} = \frac{1}{1 - t(\quad \searrow \quad + \nearrow + \quad \langle \quad)}$$

Even part $E(x, y; t) = \frac{D(x,y;t)+D(x,y;-t)}{2}$ is rational.

GF of N-bridges is algebraic

$$E(x, y; t) - [x^{>0}]E(x, y; t) - [y^{<0}]E(x, y; t)$$

Probability for a random N-walk of length $n$ to be an N-bridge

$$\frac{1 + (-1)^n}{2} \left( 1 - \frac{2\sqrt{2}}{\sqrt{\pi}} \frac{(8/9)^{n/2}}{\sqrt{n}} + O\left( \frac{(8/9)^{n/2}}{n^{3/2}} \right) \right).$$

# Dyck N-bridges

Generating function of Dyck N-walks

$$D(x, y; t) = \sum_{w} x^{\min(w)} y^{\max(w)} t^{|w|} = \frac{1}{1 - t(x^{-1}y^{-1} + xy + x^{-1}y)}$$

Even part $E(x, y; t) = \frac{D(x,y;t)+D(x,y;-t)}{2}$ is rational.

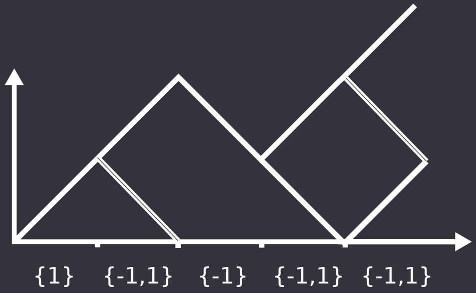
GF of N-bridges is algebraic

$$E(x, y; t) - [x^{>0}]E(x, y; t) - [y^{<0}]E(x, y; t)$$

Probability for a random N-walk of length $n$ to be an N-bridge

$$\frac{1 + (-1)^n}{2} \left( 1 - \frac{2\sqrt{2}}{\sqrt{\pi}} \frac{(8/9)^{n/2}}{\sqrt{n}} + O\left( \frac{(8/9)^{n/2}}{n^{3/2}} \right) \right).$$

# Dyck N-meanders



# Dyck N-excursions

Extremum points reachable by meanders: $\min^+(w)$, $\max^+(w)$.

$$D^+(x, y; t) := \sum_w x^{\min^+(w)} y^{\max^+(w)} t^{|w|}$$

N-meanders decomposition

$$\begin{aligned}
D^+(x, y; t) = 1 &+ t(D^+(x, y; t) - D^+(0, y; t)) \left( \searrow + \nearrow + \substack{\nearrow \\ \searrow} \right) \\
&+ t(D^+(0, y; t) - D^+(0, 0; t)) \left( \searrow + \nearrow + \substack{\nearrow \\ \searrow} \right) \\
&+ tD^+(0, 0; t) \left( \nearrow + \substack{\nearrow \\ \searrow} \right).
\end{aligned}$$

After two applications of the kernel method, algebraic GF, probability for a random N-walk of length $n$ to be an N-excursions

$$\frac{1 + (-1)^n}{2} \left( \frac{1}{4} + \sqrt{2} \frac{(8/9)^{n/2}}{\sqrt{\pi n^3}} + O\left( \frac{(8/9)^{n/2}}{n^{5/2}} \right) \right).$$

# Dyck N-excursions

Extremum points reachable by meanders: $\min^+(w)$, $\max^+(w)$.

$$D^+(x,y;t) := \sum_w x^{\min^+(w)} y^{\max^+(w)} t^{|w|}$$

N-meanders decomposition

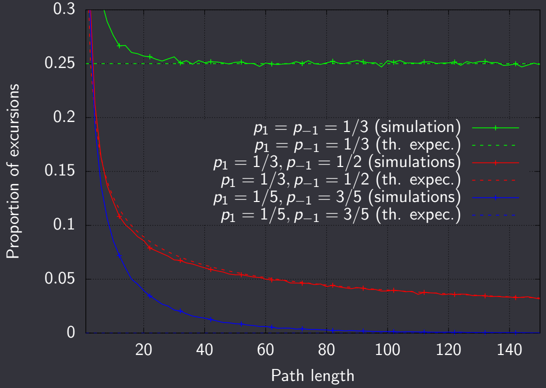
$$\begin{aligned} D^+(x,y;t) = 1 &+ t(D^+(x,y;t) - D^+(0,y;t))\left(x^{-1}y^{-1} + xy + x^{-1}y\right) \\ &+ t(D^+(0,y;t) - D^+(0,0;t))\left(xy^{-1} + xy + xy\right) \\ &+ tD^+(0,0;t)\,(xy + xy)\,. \end{aligned}$$

After two applications of the kernel method, algebraic GF, probability for a random N-walk of length $n$ to be an N-excursions

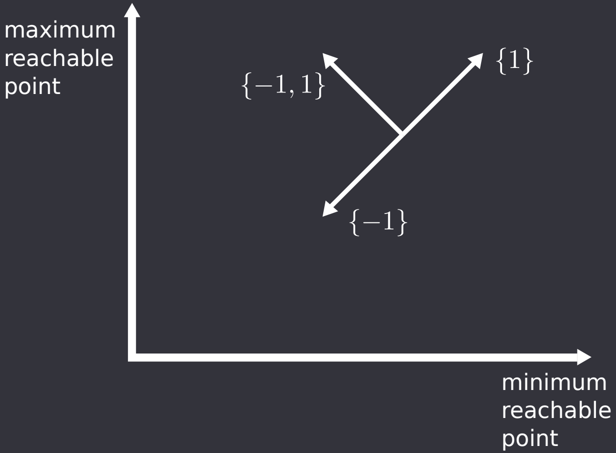
$$\frac{1+(-1)^n}{2}\left(\frac{1}{4} + \sqrt{2}\frac{(8/9)^{n/2}}{\sqrt{\pi n^3}} + O\left(\frac{(8/9)^{n/2}}{n^{5/2}}\right)\right).$$

# More precise asymptotic and numerical results

Probabilities of the N-steps: $p_{-1}$, $p_1$, $p_{-1,1} = 1 - p_{-1} - p_{-1,1}$.



# Dyck N-walks and walks in the quarter plane



# Motzkin N-walks

N-steps 

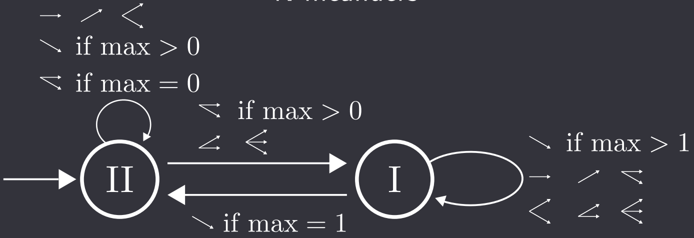
The reachable point pattern changes with the N-steps used.

Theorem: the GF of N-bridges and N-excursions are algebraic.

# Motzkin N-walks

N-walks


N-meanders



# Additive combinatorics

Sumset operator $A + B := \{a + b \mid a \in A,\ b \in B\}$.

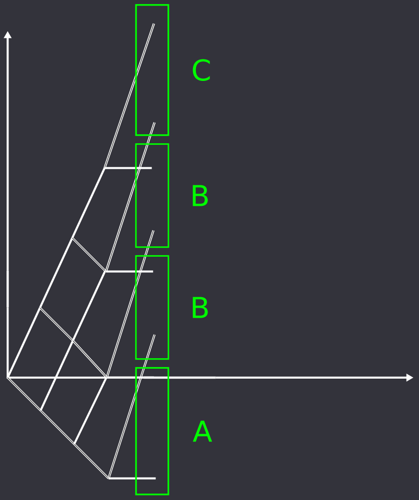
The set of reachable points of the N-walk $(w_0, \ldots, w_{n-1})$ is the sumset of its N-steps $w_0 + \ldots + w_{n-1}$.

Given an N-step family $S$, the monoid generated by $(S, +)$ describes all possible reachable sets for N-walks on $S$.

For N-meanders, consider instead $A +_{\geq 0} B := (A + B) \cap \mathbb{Z}_{\geq 0}$.

# General N-walks

$\{-1, 2\}, \{-1, 2\}, \{-1, 2\}, \{0, 3\}$




An N-walk with those reachable points is of type $(A, B, C)$.

# General N-walks

Theorem: given an N-step set $S$, there is a finite set of types $(A_i, B_i, C_i)$ such that any N-walk on $S$ has a type $(A_i, B_i, C_i)$.

Proof relies on numerical semigroup tools, and involves the Fröbenius number. Given an integer set $H$, it is the largest integer $f$ such that Equation

$$\sum_{h \in H} r_h h = f$$

has no solution $r_h \in \mathbb{Z}_{\geq 0}$.

Theorem: for any N-step set, the GF of N-walks is rational, the GF of N-bridges is algebraic.

Conjecture: the GF of N-excursions is algebraic (on the long version).

# Conclusion

Simplified model linked to discrete walks and additive combinatorics.

Generalization to series-parallel graphs instead of lines?

Generalization to several protocols?