Brick wall excursions: combinatorial interpretation of Random Flight moments

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(joint with Sergey Kirgizov and Michael Wallner)

Lattice Path Combinatorics 2026

TU Wien

July 23, 2026

Short random walks (Random Flights)

- d is the dimension,
- $\nu = \frac{d}{2} - 1$,
- $m = \#$ steps,
- A_k is a random step, $|A_k| = 1$.

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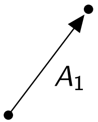

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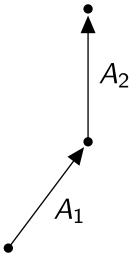

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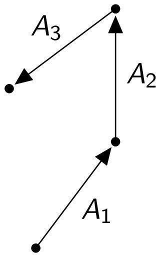

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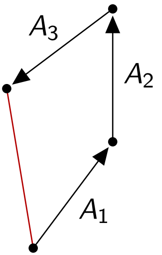

Key object: moments $W_m(\nu, n) = \mathbb{E}\Big(|A_1 + \ldots + A_m|^n\Big)$

- Fact: for any $m, n \in \mathbb{Z}_{\geqslant 0}$,
  $W_m(0, 2n)$ and $W_m(1, 2n)$ are integers. Interpretation?

Matrix form for $d=2$ $(\nu=0)$. Example: $m=2$

Fact:

$$W_m(0,2n) = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}\,, \qquad \text{where} \quad M = \left(\binom{i}{j}^2 \right)_{i,j\geqslant 0}$$

Example: $m=2,$ $W_2(0,2n) = \binom{2n}{n}$ $W_2(0,4)=6:$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & \cdots \\ 1 & 9 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \to & 1 \\ \to & 2 \\ \to & 6 \\ \to & 20 \\ \vdots & \vdots \end{matrix}$$

$UUDD$
$UDUD$
$UDDU$
$DUUD$
$DUDU$
$DDUU$

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$$M^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 6 & 8 & 1 & 0 & \cdots \\ 20 & 46 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{array}{cc} \to & 1 \\ \to & 3 \\ \to & 15 \\ \to & 77 \\ \vdots & \vdots \end{array}$$

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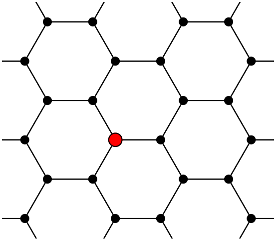

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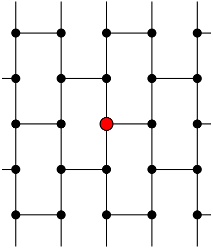

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Interpretation for $d = 2$ $(\nu = 0)$

Let $A_k \in \mathbb{C}, \quad |A_k| = 1 \qquad (k = 1, \ldots, m).$

$$
\begin{aligned}
W_m(0, 2n) \quad &= \quad \mathbb{E}\left|A_1 + \ldots + A_m\right|^{2n} \\
&\overset{(1)}{=} \quad \mathbb{E}\left(\left(A_1 + \ldots + A_m\right)\left(A_1^{-1} + \ldots + A_m^{-1}\right)\right)^n \\
&\overset{(2)}{=} \quad \left[A_1^0 \ldots A_m^0\right]\left(\left(A_1 + \ldots + A_m\right)\left(A_1^{-1} + \ldots + A_m^{-1}\right)\right)^n
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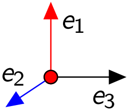
1. $1 = |A_k|^2 = A_k \bar{A}_k \qquad \Rightarrow \qquad A_k^{-1} = \bar{A}_k$
2. $\mathbb{E}\left(A_1 A_2^{-1} A_3 A_1^{-1} A_3 A_2^{-1}\right) = \mathbb{E}\left(A_2^{-2} A_3^2\right) = \mathbb{E}\left(A_2^{-2}\right) \cdot \mathbb{E}\left(A_3^2\right) = 0$

Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

$$\mathbb{E}\left(\left(\begin{array}{ccccc} A_1 & + & A_2 & + & A_3 \\ \downarrow & & \downarrow & & \downarrow \\ e_1 & & e_2 & & e_3 \end{array}\right)\left(\begin{array}{ccccc} A_1^{-1} & + & A_2^{-1} & + & A_3^{-1} \\ \downarrow & & \downarrow & & \downarrow \\ -e_1 & & -e_2 & & -e_3 \end{array}\right)\right)^n$$



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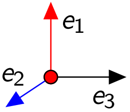
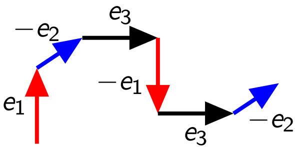
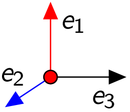




$$A_1 A_2^{-1} A_3 A_1^{-1} A_3 A_2^{-1}$$

Interpretation for $d = 2$ $(\nu = 0)$ and $m = 3$

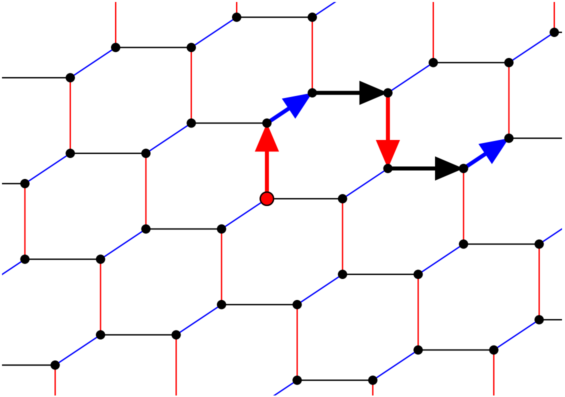
$$\mathbb{E}\Big(\big(\underset{\displaystyle e_1}{\underset{\downarrow}{A_1}} + \underset{\displaystyle e_2}{\underset{\downarrow}{A_2}} + \underset{\displaystyle e_3}{\underset{\downarrow}{A_3}}\big)\big(\underset{\displaystyle -e_1}{\underset{\downarrow}{A_1^{-1}}} + \underset{\displaystyle -e_2}{\underset{\downarrow}{A_2^{-1}}} + \underset{\displaystyle -e_3}{\underset{\downarrow}{A_3^{-1}}}\big)\Big)^n$$



terms

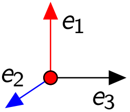
$\downarrow$

paths



Interpretation for $d = 2$ $(\nu = 0)$ and $m = 3$

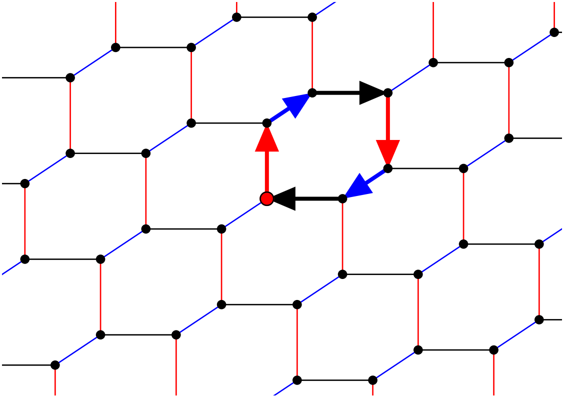
$$\mathbb{E}\Big(\big(\underset{\substack{\downarrow\\ e_1}}{A_1} + \underset{\substack{\downarrow\\ e_2}}{A_2} + \underset{\substack{\downarrow\\ e_3}}{A_3}\big)\big(\underset{\substack{\downarrow\\ -e_1}}{A_1^{-1}} + \underset{\substack{\downarrow\\ -e_2}}{A_2^{-1}} + \underset{\substack{\downarrow\\ -e_3}}{A_3^{-1}}\big)\Big)^n$$



constant terms

$\downarrow$

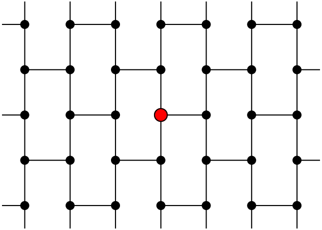
closed paths



Matrix form revisited for $d = 2$ ($\nu = 0$) and $m = 3$

Paths $\leftrightarrow$ words on $\{U, D, R, L\}$:

- R on odd positions,
- L on even positions,
- $2n = \#$ steps,



$$\#\{\text{closed paths of size } n\} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{n-k}\binom{2k}{k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 \sum_{\ell=0}^{k} \binom{k}{\ell}^2 = \sum_{\ell=0}^{n} [M^2]_{n\ell}$$

$n = 5$

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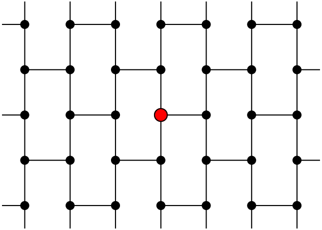

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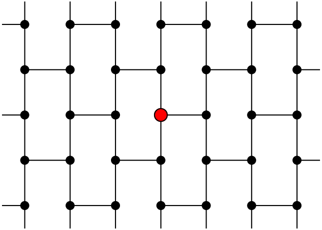

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$n = 5$ R R R $k = 2$

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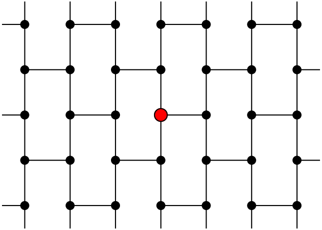

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$n = 5$ R L L L R R $k = 2$

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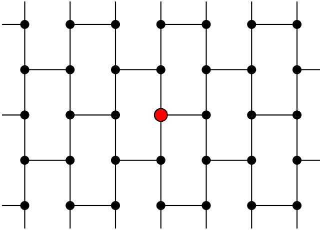

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$n = 5$ R L D L U L R U R D $k = 2$

Matrix form revisited for $d = 2$ $(\nu = 0)$ and $m = 3$

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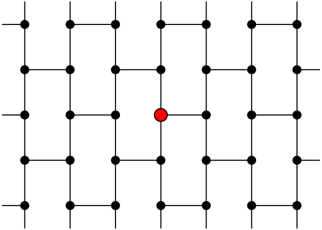

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$n = 5$ $\quad$ R L D L U L R U R D $\quad$ $k = 2$

Matrix form revisited for $d = 2$ ($\nu = 0$) and $m = 3$

Paths $\leftrightarrow$ words on $\{U, D, R, L\}$:

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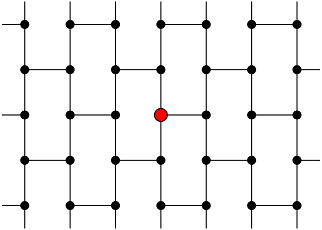

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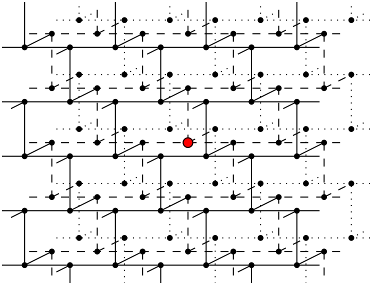
$n = 5$ R L D L U L R U R D $k = 2$

General construction for $d = 2$ ($\nu = 0$)

Paths $\leftrightarrow$ words on $\{U, D, R_1, L_1, \dots, R_{m-2}, L_{m-2}\}$:

- R_s on odd positions,
- L_s on even positions.



- $2n = \#$ steps,
- $k_0 = \#U = \#D$,
- $k_s = \#R_s = \#L_s$.

$$
\begin{aligned}
W_m(0, 2n) &= \sum_{k_0 + \ldots + k_{m-2} = n} \binom{n}{k_{m-2}}^2 \binom{n - k_{m-2}}{k_{m-3}}^2 \ldots \binom{k_1 + k_0}{k_0}^2 \binom{2k_0}{k_0} \\
&= \sum_{k_0 + \ldots + k_{m-2} = n} \binom{k_0 + \ldots + k_{m-2}}{k_0 + \ldots + k_{m-3}}^2 \ldots \binom{k_0 + k_1}{k_0}^2 \sum_{\ell=0}^{n} \binom{k_0}{\ell}^2 \\
&= \sum_{\ell, r_1, \ldots, r_{m-2} = 0}^{n} M_{n r_{m-2}} \ldots M_{r_2 r_1} M_{r_1 \ell} = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}
\end{aligned}
$$

Summary for $d = 2$ ($\nu = 0$)

- We consider moments $W_m(0, 2n) = \mathbb{E}\Big(|A_1 + \ldots + A_m|^n\Big)$, where $A_k \in \mathbb{C}$, $|A_k| = 1$ $(k = 1, \ldots, m)$.

- $W_m(0, 2n)$ is the constant term in $\Big(\big(A_1 + \ldots + A_m\big)\big(A_1^{-1} + \ldots + A_m^{-1}\big)\Big)^n$.

- Thus, $W_m(0, 2n)$ can be interpreted as the number of closed paths of length $2n$ on a specific (brick wall) $(m-1)$-dimensional lattice.

- In particular,

$$W_m(0, 2n) = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}\,, \qquad \text{where} \quad M = \left(\binom{i}{j}^2\right)_{i,j\geqslant 0}$$

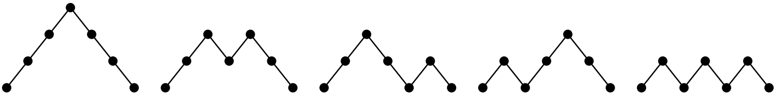
Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

Fact:

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \left(t_{i+1,j+1}\right)_{i,j\geqslant 0}$$

Here, $t_{i,j}$ are the **Narayana numbers**, *i.e.*

$$t_{i,j} = \#\{\text{Dyck paths of length } i \text{ with } j \text{ peaks}\}$$



$\underbrace{}_{t_{3,1}=1}$ $\underbrace{}_{t_{3,2}=3}$ $\underbrace{}_{t_{3,3}=1}$

Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

Fact:

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \left(t_{i+1,\, j+1}\right)_{i,j \geqslant 0}$$

Example: $m = 2$, $W_2(1, 2n) = C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & \cdots \\ 1 & 6 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \to & 1 \\ \to & 2 \\ \to & 5 \\ \to & 14 \\ \vdots & \vdots \end{matrix}$$

$W_2(1, 4) = 5$:

UUUDDD
UUDUDD
UUDDUD
UDUUDD
UDUDUD

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$W_2(1, 4) = 5$:

$UUUDDD$
$UUDUDD$
$UUDDUD$
$UDUUDD$
$UDUDUD$

Bijective lemma

D_n is words on $\{R, L, O\}$ such that:

- R on odd positions,
- L on even positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$,
- $2n = \#$ letters.

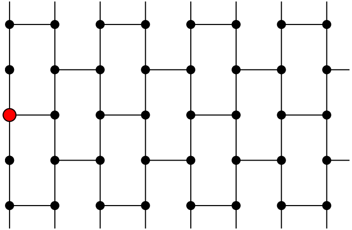
$$D_n \quad = \quad \sum_{k=0}^{n} t_{n+1,\,k+1}$$

R O O L O O R L $\qquad 2k = \#O$

Bijective lemma and its consequences

Paths, half-plane:

- R on odd positions,
- L on even positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$,
- $2n = \#$ steps.



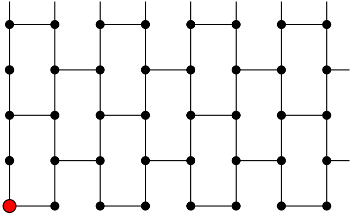
$$\#\{\text{closed paths}\} = \sum_{k=0}^{n} t_{n+1,\,k+1} \binom{2k}{k}$$

$O \rightsquigarrow U, D$ R D U L U D R L $k = \#U$ $k = \#D$

Bijective lemma and its consequences

Paths, quarter-plane:

- R on odd positions,
- L on even positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$,
- $2n = \#$ steps.



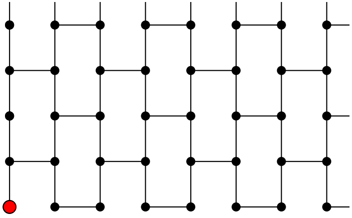
$$\#\{\text{closed paths}\} \quad = \quad \sum_{k=0}^{n} t_{n+1,\,k+1} C_k$$

$O \rightsquigarrow U, D$ R U D L U D R L $k = \#U$ $k = \#D$

Bijective lemma and its consequences

Paths, shifted quarter-plane:

- R on even positions,
- L on odd positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$,
- $2n+2 = \#$ steps.



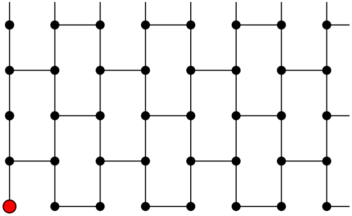
$$\#\{\text{closed paths}\} = \sum_{k=0}^{n} t_{n+1,\,k+1} C_{k+1}$$

U R U D L U D R L D

Bijective lemma and its consequences

Paths, shifted quarter-plane:

- R on even positions,
- L on odd positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$,
- $2n + 2 = \#$ steps.



$$\#\{\text{closed paths}\} = \sum_{k=0}^{n} t_{n+1,\,k+1}\, C_{k+1}$$

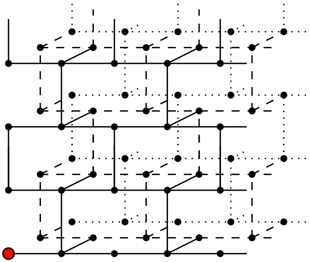
$$= \sum_{k=0}^{n} t_{n+1,\,k+1} \sum_{\ell=0}^{k} t_{k+1,\,\ell+1} = \sum_{\ell=0}^{n} [N^2]_{n\ell}$$

U R U D L U D R L D

General construction for $d=4$ ($\nu=1$)

Paths $\leftrightarrow$ words on $\{U, D, R_1, L_1, \ldots, R_{m-2}, L_{m-2}\}$:

- after removing R_t and L_t, $t > s$:
  - R_s on even positions,
  - L_s on odd positions,
- in each prefix: $\#U \geqslant \#D$ and $\#R_s \geqslant \#L_s$.



- $2n = \#$ steps,
- $k_0 = \#U = \#D$,
- $k_s = \#R_s = \#L_s$.

$$\begin{aligned}
W_4(1,2n) &= \sum_{k_0+k_1+k_2=n} t_{n+1,\,k_1+k_0+1} \cdot t_{n-k_2+1,\,k_0+1} \cdot C_{k_0+1} \\
&= \sum_{k_0+k_1+k_2=n} t_{n+1,\,k_1+k_0+1} \cdot t_{k_1+k_0+1,\,k_0+1} \sum_{\ell=0}^{n} t_{k_0+1,\,\ell+1} \\
&= \sum_{\ell,r_1,r_2=0}^{n} N_{nr_2} N_{r_2 r_1} N_{r_1 \ell} = \sum_{\ell=0}^{n} [N^3]_{n\ell}
\end{aligned}$$

Summary for $d = 4$ ($\nu = 1$)

- We consider moments $W_m(1, 2n) = \mathbb{E}\Big(|A_1 + \ldots + A_m|^n\Big)$, where $A_k \in \mathbb{R}^4$, $|A_k| = 1 \quad (k = 1, \ldots, m)$.
- It is known that,

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = (t_{i+1, j+1})_{i, j \geqslant 0}$$

- From this, $W_m(1, 2n)$ can be interpreted as the number of closed paths of length $2n$ on a specific m-dimensional lattice.
- Question. Can we obtain the above result directly? (one could expect the use of quaternions)

Perspectives

- Dimension $d = 4$: quaternions?
  - Free probability.
  - Matrix integrals.
- Higher dimensions, $d = 2(\nu + 1)$?
  - Matrix entries are rational.
  - $\binom{2\nu-1}{2} W_m(\nu; 2n)$ are integers?
- Odd moments are transcendental.
- Odd dimensions?
  - Moments are rational.
  - Structure is hidden.

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(link to the paper)

Thank you for your attention!

Literature

Borwein J.M., Straub A., Vignat C.
Densities of short uniform random walks in higher dimensions
J. Math. Anal. Appl., 437(1): pp. 668–707, 2016.

Kirgizov S., Nurligareev K., Wallner M.
Brick wall excursions: combinatorial interpretation of Random Flight moments
ArXiv: 2605.16715, 2026.

Bijective lemma

The Narayana number

$$t_{n+1,\,k+1} = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}$$

counts:

words on $\{R, L, O\}$ such that

- R on odd positions,
- L on even positions,
- in each prefix, $\#R \geqslant \#L$,
- $\#R = \#L = n - k$,
- $\#O = 2k$.

R L R O R O O L O L

Bijective lemma

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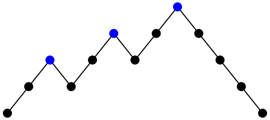
R L R O R O O L O L

Dyck paths:

- of size $(n+1)$,
- with $(k+1)$ peaks.



Bijective lemma

The Narayana number

$$t_{n+1,\,k+1} = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1} = t_{n+1,\,n-k+1}$$

counts:

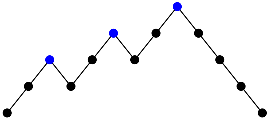
words on $\{R, L, O\}$ such that

- R on odd positions,
- L on even positions,
- in each prefix, $\#R \geqslant \#L$,
- $\#R = \#L = k$,
- $\#O = 2(n-k)$.

Dyck paths:

- of size $(n+1)$,
- with $(k+1)$ peaks.

O O R O R O O L O L $\rightsquigarrow$



Bijection, part I

1. Take a word.

R O O O R O O L O L O O R O R O O L O L

Bijection, part I

1. Take a word.
2. Identify
   - external R's and L's,

R O O O R O O L O L O O R O R O O L O L

Bijection, part I

1. Take a word.
2. Identify
   - external R's and L's,
   - external O's.

$$\underline{R\ O\ O\ O\ R\ O\ O\ L\ O\ L}\ O\ O\ \underline{R\ O\ R\ O\ O\ L\ O\ L}$$

Bijection, part I

1. Take a word.
2. Identify
   - external R's and L's,
   - external O's.
3. Add one more R and L:
   - R: just before the first external O,
   - L: in the end.

R O O O R O O L O L R O O R O R O O L O L L

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   - external R's and L's,
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4. Determine subwords $R\alpha_k L$.

$$\underbrace{\text{R O O O R O O L O L}}_{R\alpha_1 L}\ \underbrace{\text{R O O R O R O O L O L L}}_{R\alpha_2 L}$$

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1. Take a word.
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   - R: just before the first external O,
   - L: in the end.
4. Determine subwords $R\alpha_k L$.
5. Treat each subword separately.

$$\underbrace{\mathrm{R\ O\ O\ O\ R\ O\ O\ L\ O\ L}}_{R\alpha_1 L}\ \underbrace{\mathrm{R\ O\ O\ R\ O\ R\ O\ O\ L\ O\ L\ L}}_{R\alpha_2 L}$$

Bijection, part II

$$\mathrm{R}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{x}\ \underline{\mathrm{R}\,\beta_1\,\mathrm{L}}\ \underbrace{\mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}}_{2i_1}\quad\ldots\quad\underbrace{\mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}}_{2i_{m-1}}\ \underline{\mathrm{R}\,\beta_m\,\mathrm{L}}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{y}\ \mathrm{L}$$

Bijection, part II

$$\mathrm{R}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{x}\ \underline{\mathrm{R}\,\beta_1\,\mathrm{L}}\ \underbrace{\mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}}_{2i_1}\quad\ldots\quad\underbrace{\mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}}_{2i_{m-1}}\ \underline{\mathrm{R}\,\beta_m\,\mathrm{L}}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{y}\ \mathrm{L}$$

$$2k = \begin{cases} x - y & \text{if } x > y \\ y - x + 2 & \text{if } x \leqslant y \end{cases} \qquad \downarrow \qquad j = \begin{cases} y & \text{if } x > y \\ x - 1 & \text{if } x \leqslant y \end{cases}$$

$$\mathrm{R}\ \underbrace{\mathrm{O}\ \mathrm{O}\ \mathrm{O}\ldots\mathrm{O}\ \mathrm{O}\ \mathrm{O}}_{j+2k+i_1+\ldots+i_{m-1}}\ \underline{\mathrm{R}\,\beta_1\,\mathrm{L}}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{i_1}\quad\ldots\quad\underbrace{\mathrm{O}\ldots\mathrm{O}}_{i_{m-1}}\ \underline{\mathrm{R}\,\beta_m\,\mathrm{L}}\ \underbrace{\mathrm{O}\ldots\mathrm{O}}_{j}\ \mathrm{L}$$

Bijection, part II

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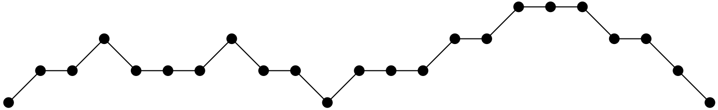
$$\downarrow$$

$$\text{R}\ \underbrace{\text{R R R}\ldots\text{R R L}}_{j+2k+i_1+\ldots+i_{m-1}}\ \underline{\text{R}\,\beta_1\,\text{L}}\ \underbrace{\text{L}\ldots\text{L}}_{i_1}\quad\ldots\quad\underbrace{\text{L}\ldots\text{L}}_{i_{m-1}}\ \underline{\text{R}\,\beta_m\,\text{L}}\ \underbrace{\text{L}\ldots\text{L}}_{j}\ \text{L}$$

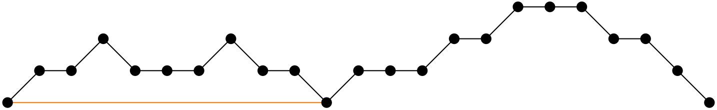
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L



Bijection, example

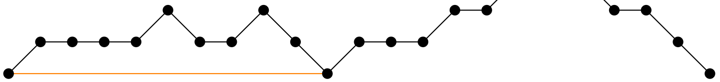
R O R L O O R L O L R O O R O R O O L O L L



Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

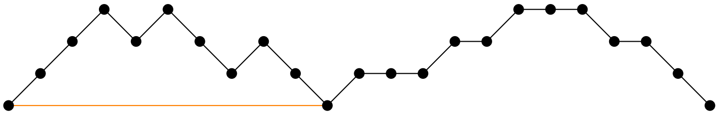
R O O O R L O R L L R O O R O R O O L O L L



Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

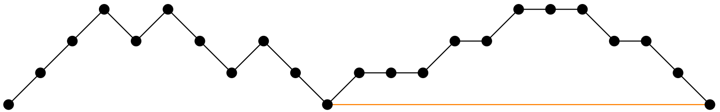
R R R L R L L R L L R O O R O R O O L O L L



Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

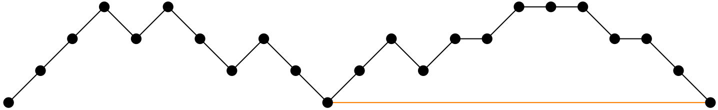


Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

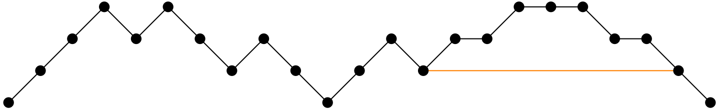


Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L



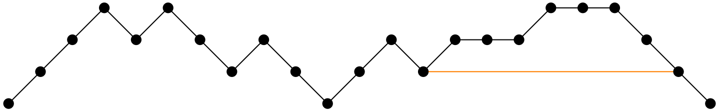
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

R R R L R L L R L L R R L R O O R O O L L L



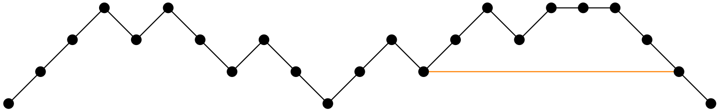
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

R R R L R L L R L L R R L R R L R O O L L L



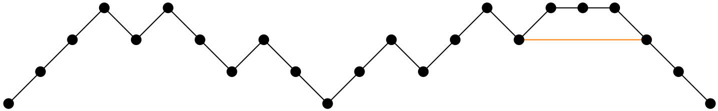
Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

R R R L R L L R L L R R L R R L R O O L L L



Bijection, example

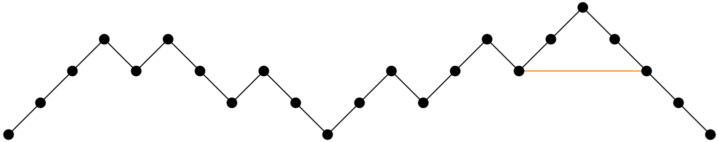
R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

R R R L R L L R L L R R L R R L R O O L L L

R R R L R L L R L L R R L R R L R R L L L L



Bijection, example

R O R L O O R L O L R O O R O R O O L O L L

R R R L R L L R L L R O O R O R O O L O L L

R R R L R L L R L L R R L R O R O O L O L L

R R R L R L L R L L R R L R R L R O O L L L

R R R L R L L R L L R R L R R L R R L L L L