A combinatorial model for the canonical join complex of alt ν-Tamari lattices

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Canonical Join Complex: Motivation

Factorization

Canonical Join Complex

L is a finite join-semidistributive lattice
Join irreducible element of L: exactly one down cover

Canonical join representation of $w \in L$:
unique "lowest" subset $A \subseteq \mathrm{JoinIrr}(L)$ satisfying $\bigvee A = w$

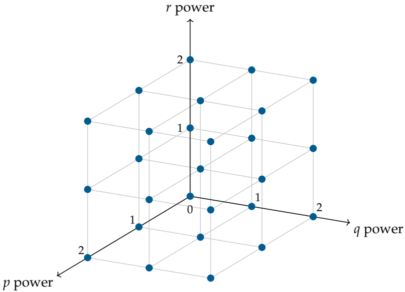
The canonical join complex is the simplicial complex:

facets $\longleftrightarrow$ canonical join representations in L

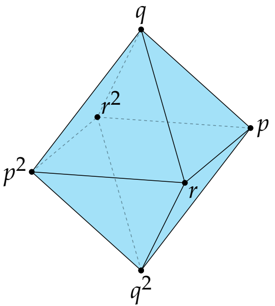
Theorem (Reading 2015)

The canonical join complex is a complex.

Example: Lattice L of divisors of $p^2q^2r^2$ for primes p, q, r



structure of L



canonical join complex of L

The canonical join complex for the lattice of divisors of $N = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ is the complete multipartite complex.

First combinatorial model

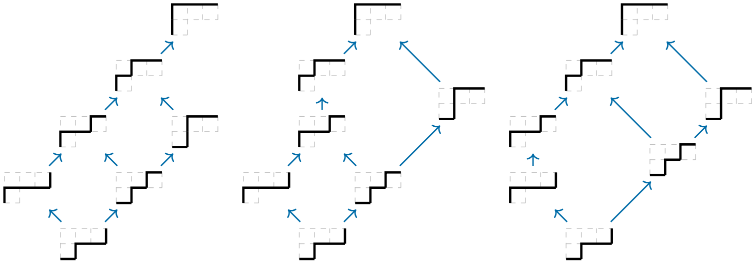
Reading provided a combinatorial model for the canonical join representation of permutations in terms of non-crossing arc diagrams.

Theorem (Reading 2015, Barnard 2020)

The canonical join complex of the Tamari lattice is realized by right non-crossing arc diagrams.

Question

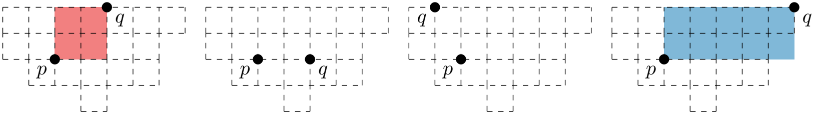
Can we give a combinatorial model for variants of the Tamari lattice?



Alt ν-Tamari lattices

Alt ν-Tamari lattice

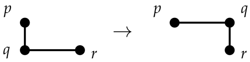
Given a stack polyomino $F_{\delta,\nu}$, the alt ν-Tamari lattice is the rotation lattice of (δ, ν)-trees.



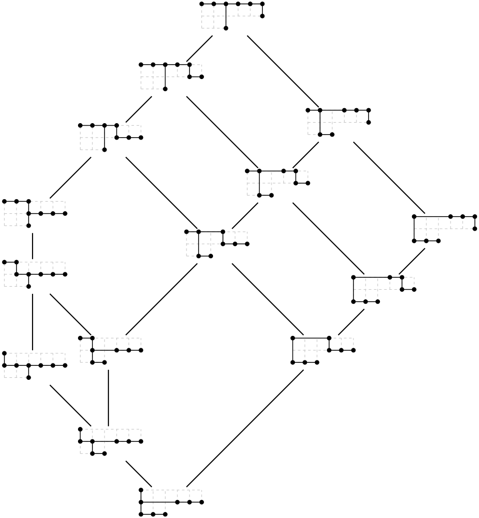
(δ, ν)-incompatible $\quad$ (δ, ν)-compatible $\quad$ (δ, ν)-compatible $\quad$ (δ, ν)-compatible

(δ, ν)-Tree: maximal collection of (δ, ν)-compatible lattice points in $F_{\delta,\nu}$.

Rotation:

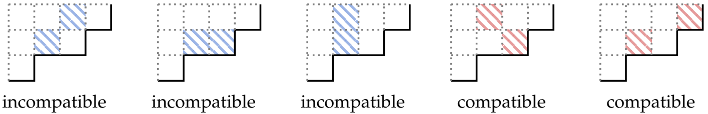


Alt ν-Tamari lattice: Example



Combinatorial Model

Compatibility relation of boxes in $F_{\delta,\nu}$:



Definition (Müller 2026)

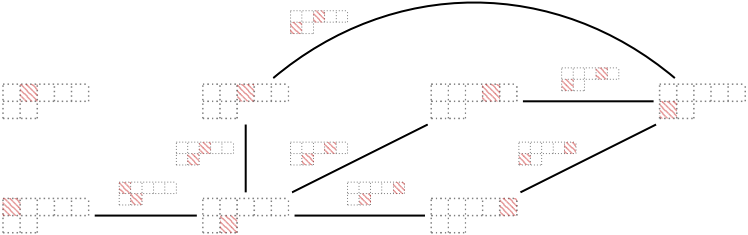
The box complex $\Delta_{\delta,\nu}$ is the simplicial complex, whose

$$\text{faces} \longleftrightarrow \text{pairwise compatible boxes in } F_{\delta,\nu}.$$

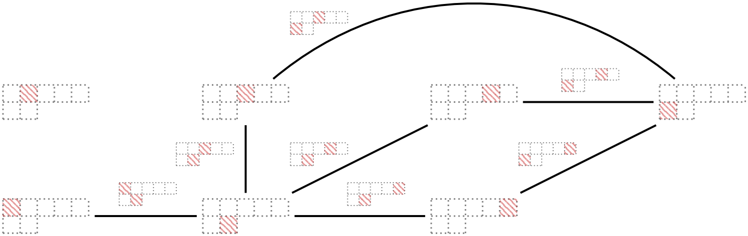
Realization

Theorem (Müller 2026)

The box complex realizes the canonical join complex of the alt ν-Tamari lattice.



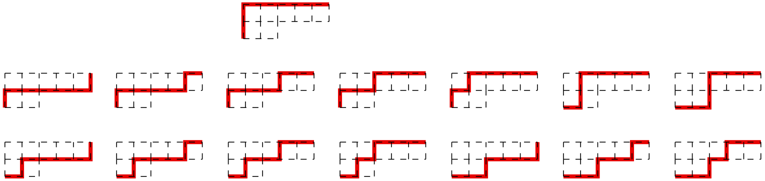
Euler characteristic: Example



Euler characteristic:

$$\chi(\Delta_{\delta,\nu}) = \sum_{i=0}^{\dim(\Delta_{\delta,\nu})} (-1)^i f_i(\Delta_{\delta,\nu}) = 7 - 7 = 0$$

ν-Narayana polynomial: Example



x^0

$7x^1$

$7x^2$

$$N_\nu(x) = 1 + 7x + 7x^2$$
$$1 - N_\nu(-1) = 1 - (1 - 7 + 7) = 0$$

Theorem (Müller 2026)

Euler characteristic: $\chi(\Delta_{\mathrm{Tam}_\nu(\delta)}) = 1 - N_\nu(-1)$.

Topology

Theorem (Müller 2026)

The canonical join complex $\Delta_{\delta,\nu}$ of alt ν-Tamari lattices is vertex decomposable. In particular, it is shellable.

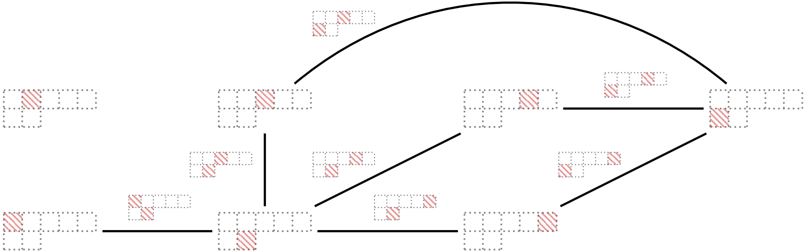
It follows from the work of Björner and Wachs that $\Delta_{\delta,\nu}$ is homotopy equivalent to a wedge of spheres.

Homology: Invariance

Theorem (Müller 2026)

For a fixed $\nu = NE^{\nu_1} \dots NE^{\nu_n}$, the number of $(n-2)$-dimensional spheres is constant for all alt ν-Tamari lattices.

Homology: Example



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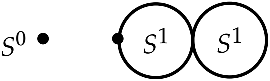
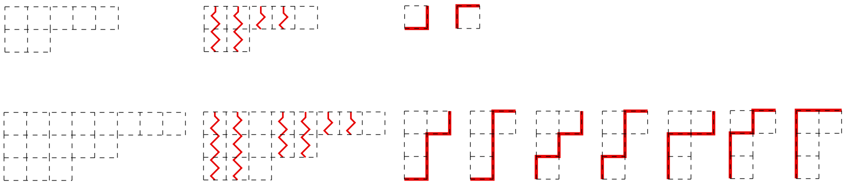


Homology

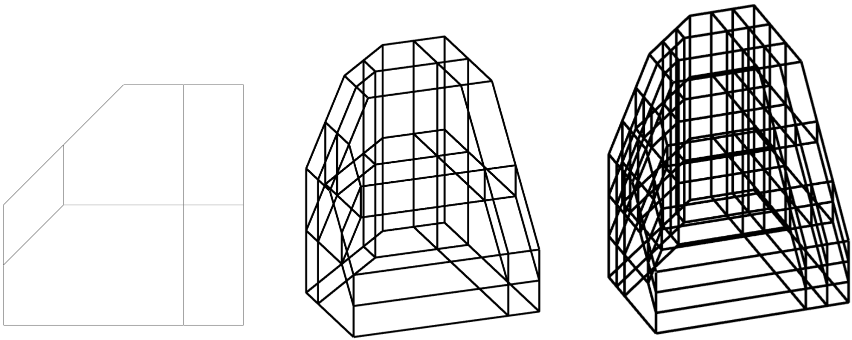
$$\nu = NE^{\nu_1} \dots NE^{\nu_n}, \qquad \nu_i \geq 2,\ 1 \leq i \leq n,$$
$$\bar{\nu} = NE^{\nu_1 - 2} \dots NE^{\nu_n - 2}$$

Theorem (Müller 2026)

$$\#\ (n-2)\text{-spheres} \quad = \quad \#\bar{\nu}\text{-Dyck paths}$$



Special case: *m*-Tamari lattice



Special case: *m*-Tamari lattice

Corollary (Müller 2026)

For $\nu = (NE^m)^n$ with $n, m \geq 2$, the number of $(n-2)$-spheres is given by the shifted Fuss-Catalan number

$$\frac{1}{(m-2)n+1}\binom{(m-1)n}{n}.$$

The box complex is a universal tool designed to make everything simpler.

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