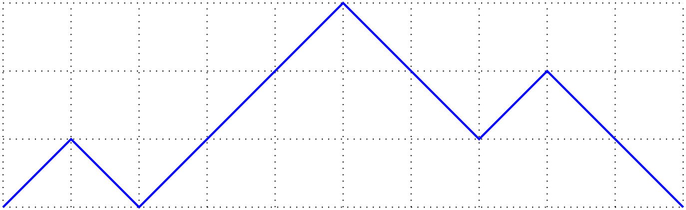
Bouncing canons

Krishna Menon – KTH Royal Institute of Technology

LPC 2026, Vienna

Canon permutations

$D =$ 

$\pi = \; 3\,5\,2\,1\,4$

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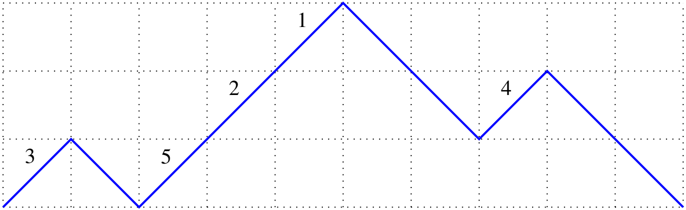
$D =$ 

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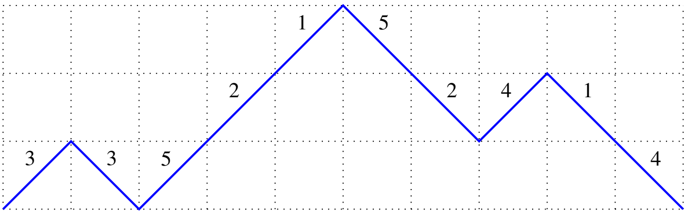
$D =$ 

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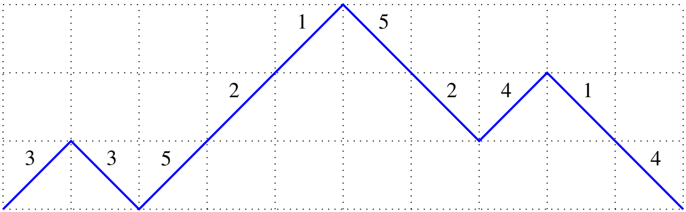
$D =$ 

$\pi = \ 3\ 5\ 2\ 1\ 4$

$$\mathrm{can}(D, \pi) = 3\ 3\ 5\ 2\ 1\ 5\ 2\ 4\ 1\ 4$$

Descents

For any list of numbers $\pi = \pi_1\pi_2\cdots\pi_n$

$$\mathrm{des}(\pi) := \#\{i \in [n-1] \mid \pi_i > \pi_{i+1}\}.$$

$$\pi = 3 \quad 1 \quad 4 \quad 1 \quad 5 \quad 9 \quad 2 \quad 6$$

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$\pi = 3$  $1 \quad 4$  $1 \quad 5 \quad 9$  $2 \quad 6$

$$\mathrm{des}(\pi) = 3$$

History

- Elizalde (2024, 2025, 2026+):
  - initiated study of descents in canon permutations
  - revealed nice structure of their descent polynomials
  - extended results to ‘higher dimensions’ through SYTs
- Beck and Deligeorgaki (2026+):
  - extended definitions and results to labeled posets
  - simpler proofs of nice structure of descent polynomials
  - introduced a generalization: *dissonant* canon permutations

Theorem (Elizalde, 2024). For $\pi \in \mathfrak{S}_n$, we have

$$\sum_{D \in \mathrm{Dyck}_n} t^{\operatorname{des}(D,\pi)} = t^{\operatorname{des}\pi} N_n(t)$$

where $N_n(t)$ is a Narayana polynomial.

What Elizalde has done: Fix π, vary D

Goal for this talk

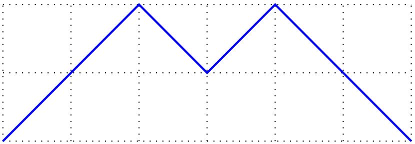
To any $D \in \mathrm{Dyck}_n$, we associate

$$C_D(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(D,\pi)}.$$

What can we say about $C_D(t)$?

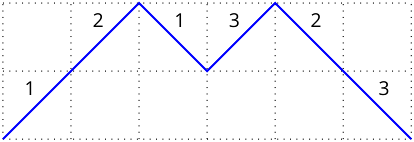
Based on joint work with Danai Deligeorgaki

Collaborator on a canon

$D =$ 

π	$\mathrm{des}(D,\pi)$
1 2 3	2
1 3 2	2
2 1 3	2
2 3 1	3
3 1 2	3
3 2 1	3

$$C_D(t) = 3t^2 + 3t^3$$

$D =$ 

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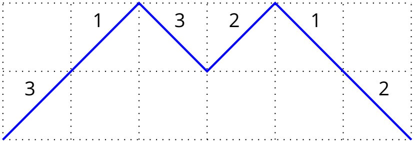
$D =$ 

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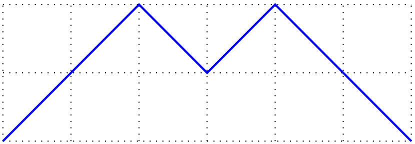
$D =$ 

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Results

- Show that $C_D(t)$ is palindromic.
- Find the degree of $C_D(t)$.
- Show that $C_D(t)$ has no internal zeroes.

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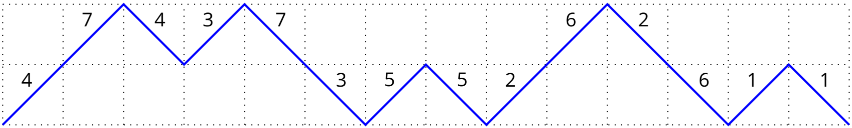
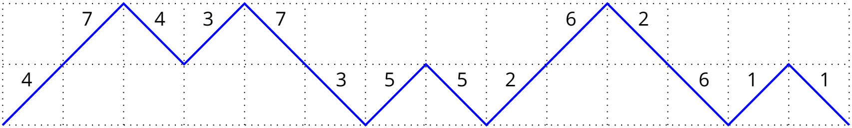
A simple observation

Adjacent terms equal $\leftrightarrow$ Low peaks



$$\operatorname{lpk} D = 2$$

$$1 \to \color{red}{7} \qquad 2 \to \color{red}{6} \quad \cdots \quad 6 \to \color{red}{2} \qquad 7 \to \color{red}{1}$$



Descents $\longrightarrow$ Ascents   Ascents $\longrightarrow$ Descents   Plateaus $\longrightarrow$ Plateaus

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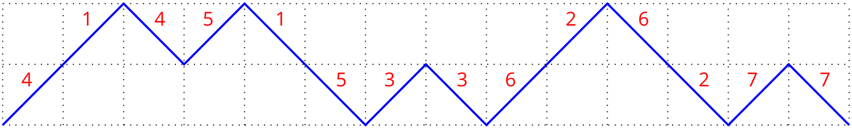

Descents ⟶ Ascents   Ascents ⟶ Descents   Plateaus ⟶ Plateaus

Palindromicity

Result. For any $D \in \mathrm{Dyck}_n$ and $i \geqslant 0$.

$$[t^i]C_D(t) = [t^{k-i}]C_D(t)$$

where $k = 2n - 1 - \mathrm{lpk}\, D$.

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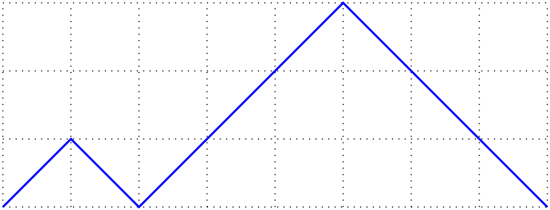
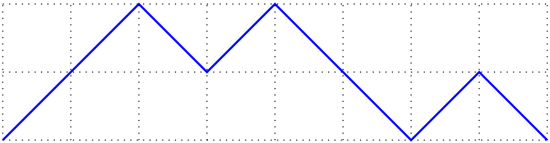
Note: Does not say anything about degree!



$$C_D(t) = t + 8t^2 + 6t^3 + 8t^4 + t^5$$

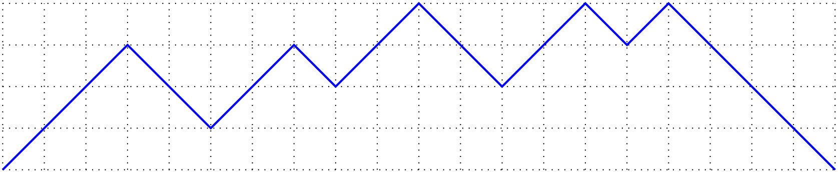


$$C_D(t) = 4t^2 + 16t^3 + 4t^4$$

Results

- ✓ Show that $C_D(t)$ is palindromic.
- Find the degree of $C_D(t)$.
- Show that $C_D(t)$ has no internal zeroes.

Main ingredient: bounce path



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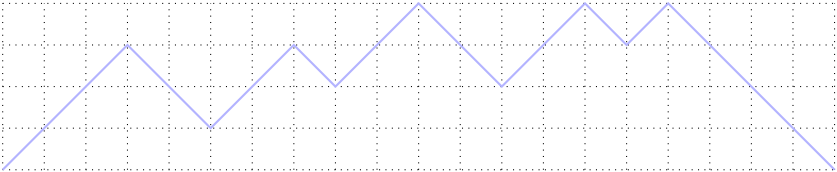

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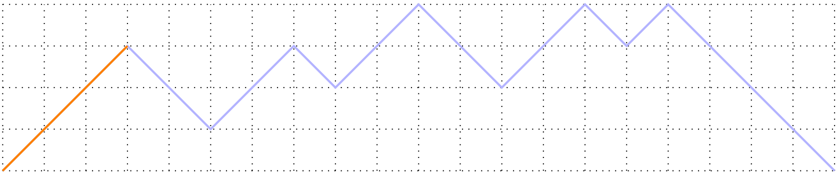

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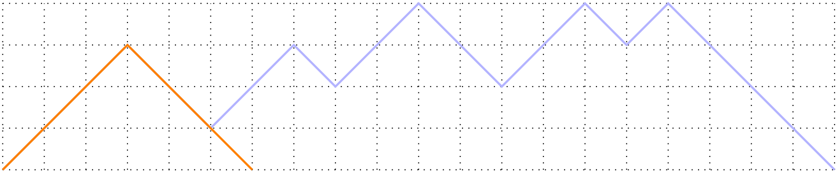

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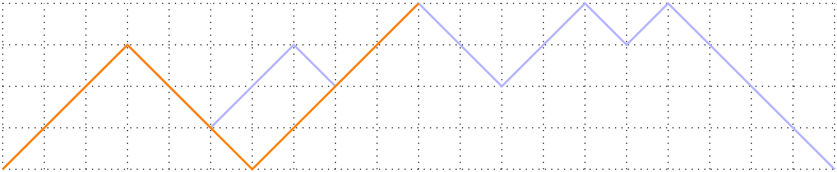

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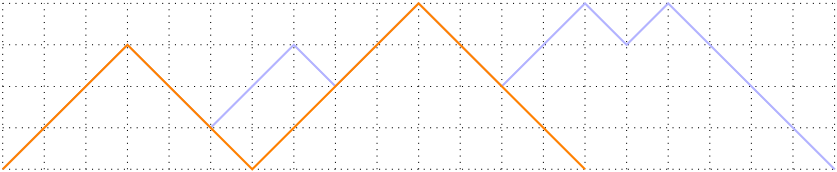

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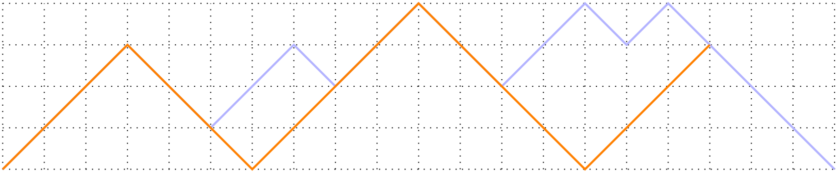

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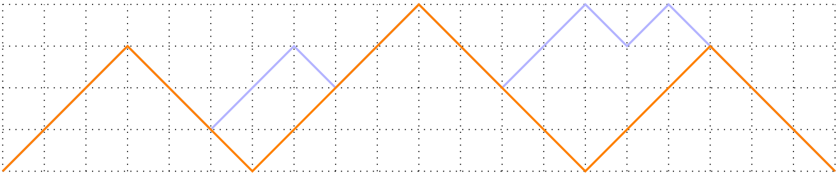

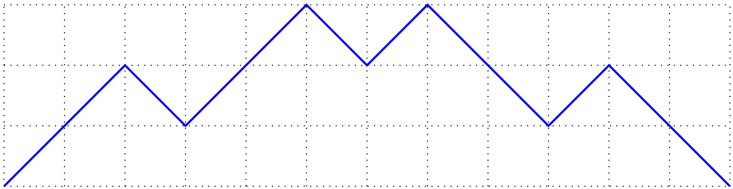

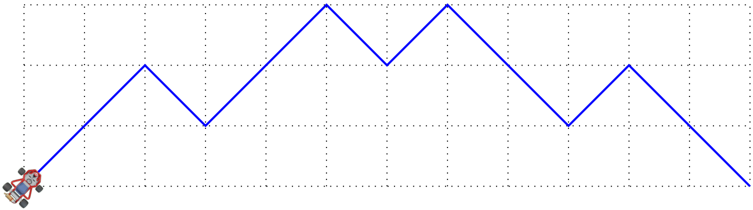

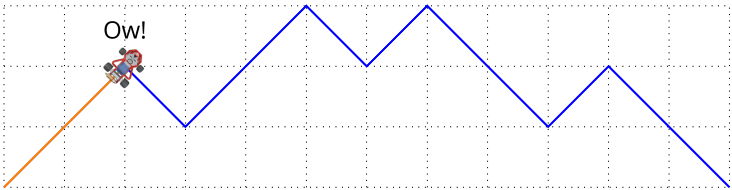
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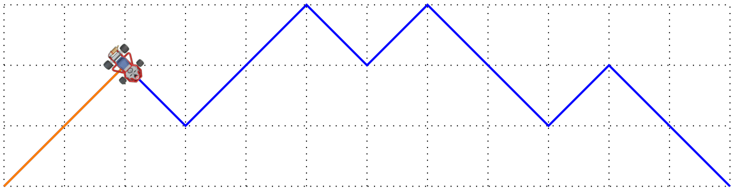

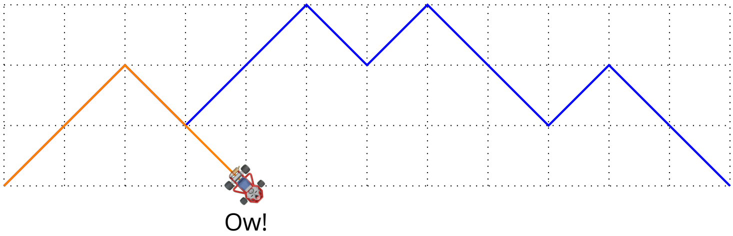
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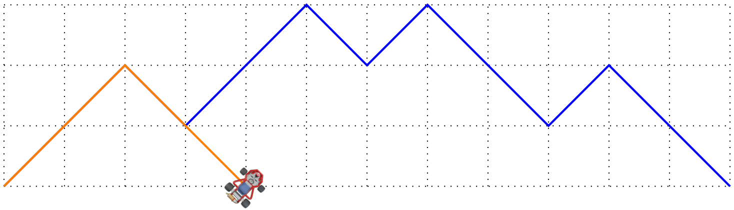

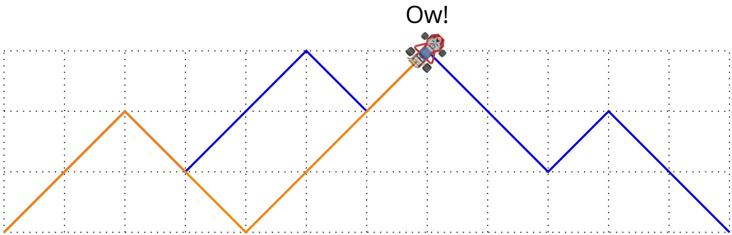
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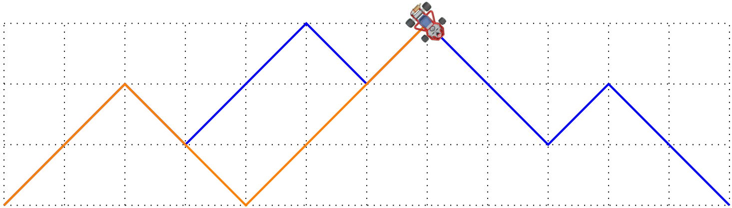

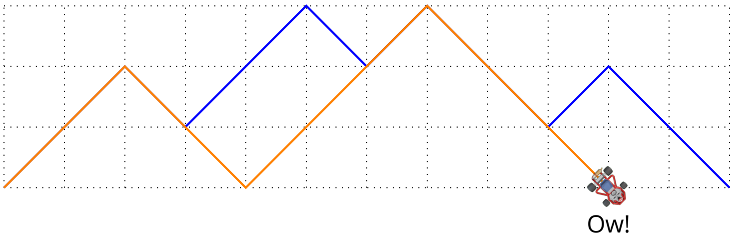
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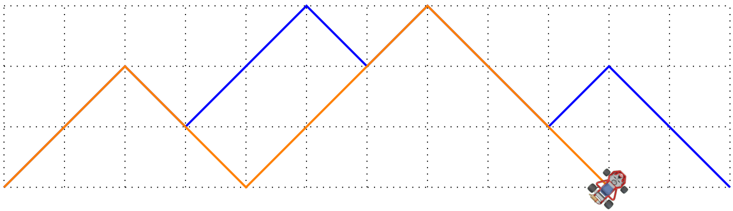

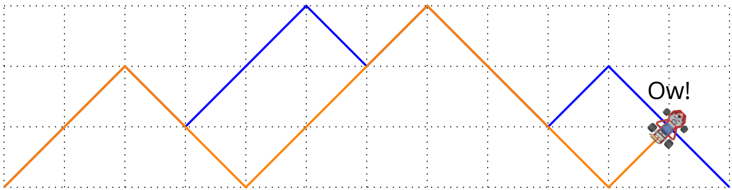
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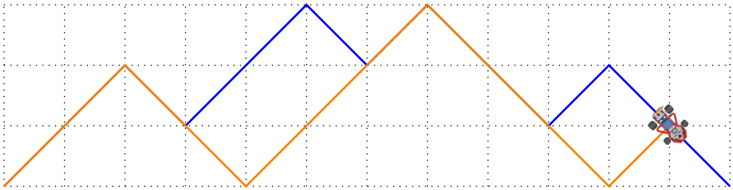

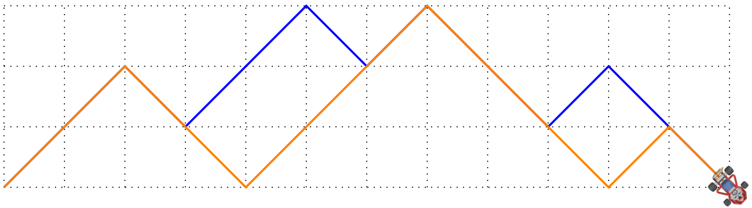

Another simple observation

Between two equal numbers, there must be a non-descent.

$$\cdots \quad a \quad \cdots \quad a \quad \cdots$$

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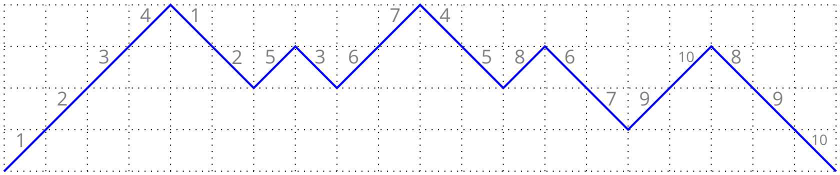
Between two equal numbers, there must be a non-descent.

$$\cdots \quad a_1 \quad \cdots \quad a_1 \quad \cdots \quad a_2 \quad \cdots \quad a_2 \quad \cdots\cdot \quad a_k \quad \cdots \quad a_k \quad \cdots$$

Lemma. If $\mathrm{can}(D, \pi)$ has a subsequence of the form above,

$$\mathrm{des}(D, \pi) \leqslant 2n - 1 - k.$$


1
2
3
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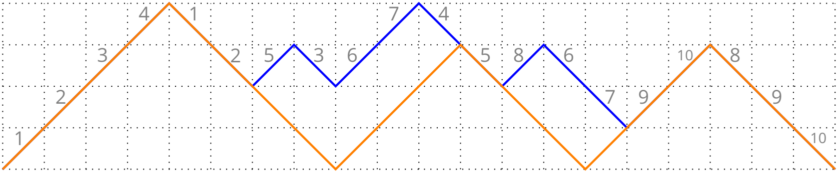
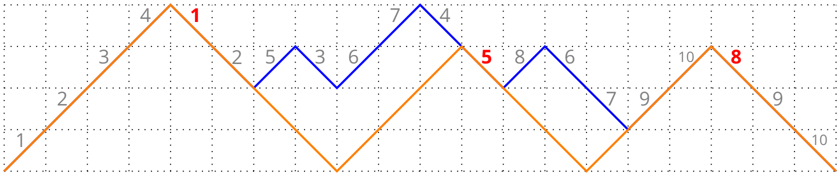
10



Peaks of the bounce path choose such a sequence!



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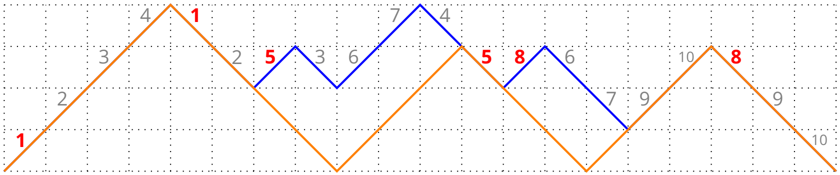

Peaks of the bounce path choose such a sequence!

Lemma. For $D \in \mathrm{Dyck}_n$ and $\pi \in \mathfrak{S}_n$, we have

$$\mathrm{des}(D, \pi) \leqslant 2n - 1 - \mathrm{pk}(\mathrm{bounce}\, D).$$

Lemma. For $D \in \mathrm{Dyck}_n$, we have

$$\deg C_D(t) \leqslant 2n - 1 - \mathrm{pk}(\mathrm{bounce}\, D).$$

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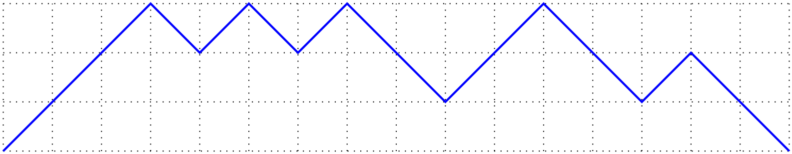
This is an equality!

How to bounce a canon

1. Bounce path labeled with decreasing permutation.

2. Swap valleys, relabeling to maintain descent structure.



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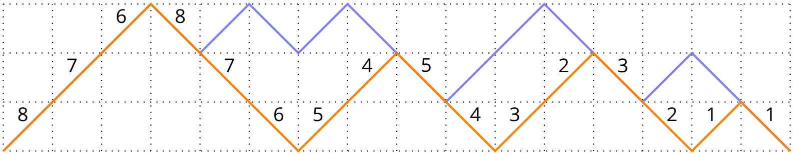

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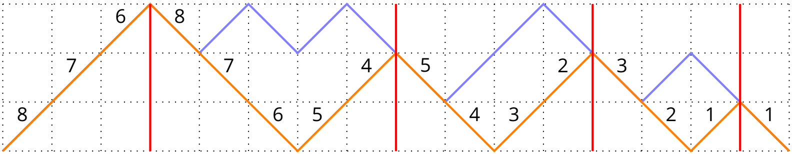

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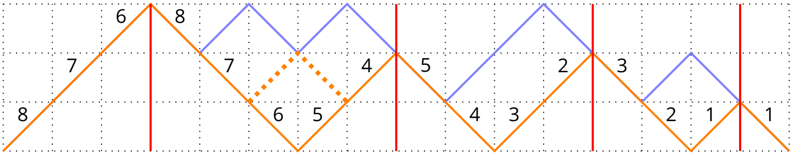

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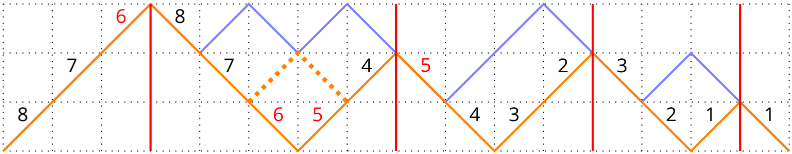

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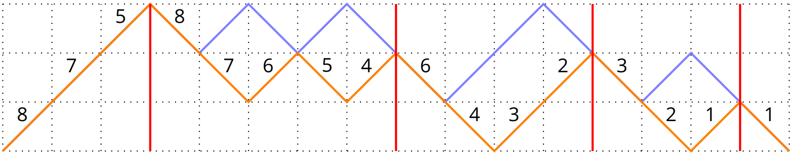

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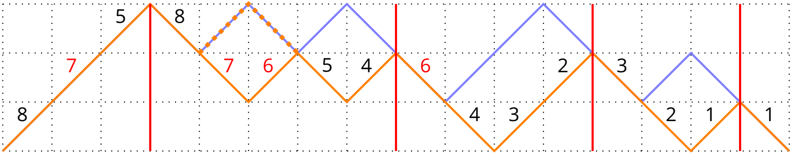

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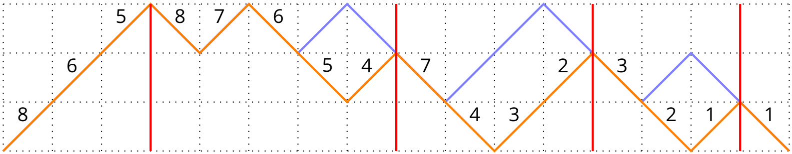

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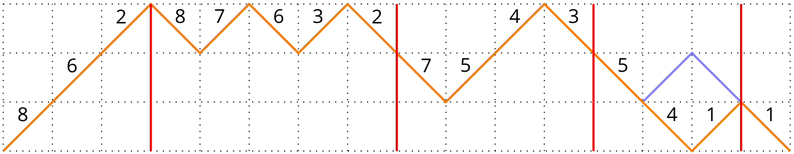

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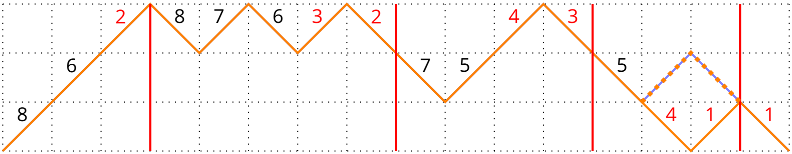

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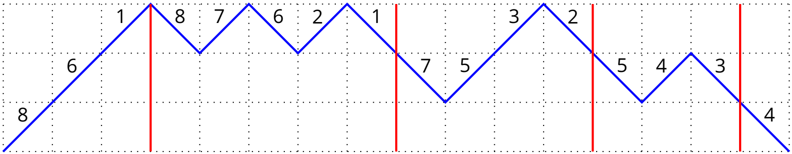


Theorem. For any $D \in \mathrm{Dyck}_n$, we have

$$\deg C_D(t) = 2n - 1 - \mathrm{pk}(\mathrm{bounce}\, D).$$

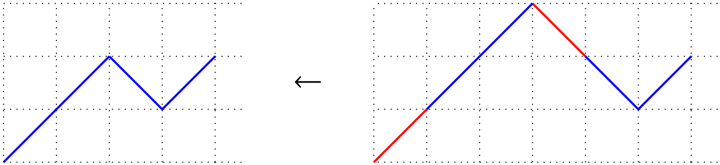
Results

- ✓ Show that $C_D(t)$ is palindromic.
- ✓ Find the degree of $C_D(t)$.
- Show that $C_D(t)$ has no internal zeroes.

- $C_D(t)$ should not be something like

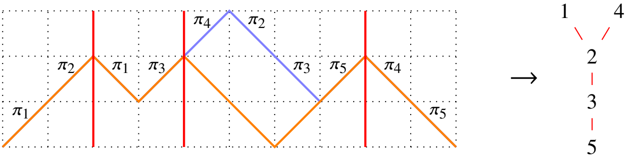
$$C_D(t) = 2t^3 + 5t^4 + 0t^5 + 5t^6 + 2t^7.$$

- Idea: Induction, delete the first up and down step.

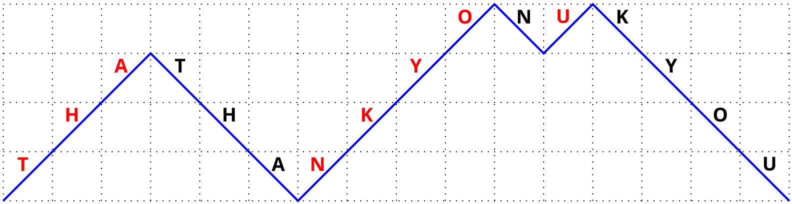


What else?

- Descent sets of maximizers via generalized bounce paths.
- Maximizer permutations as linear extensions of posets.

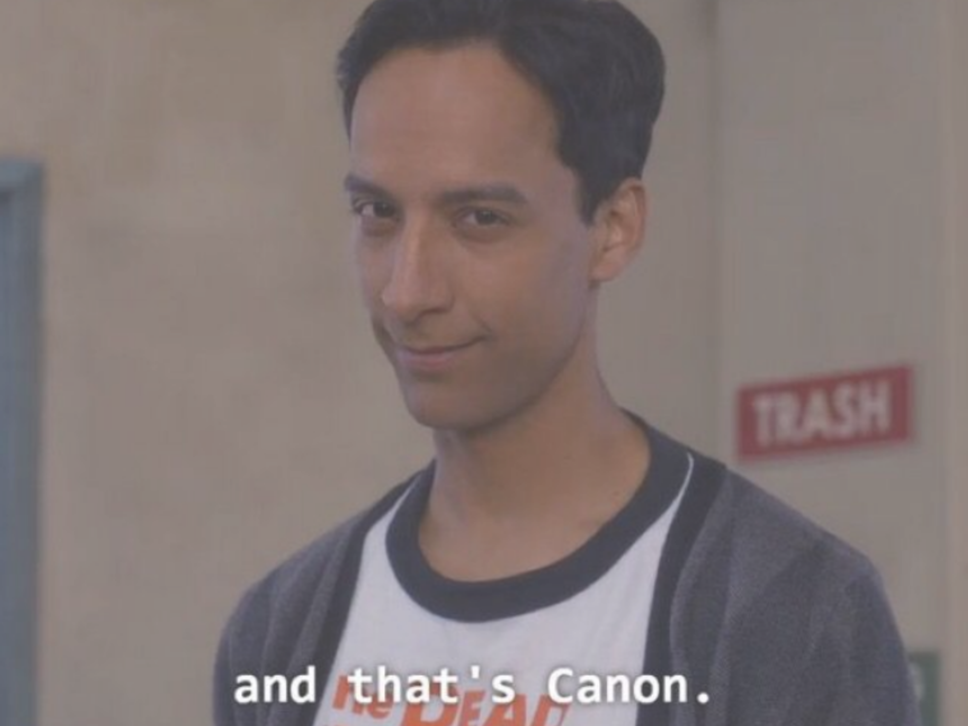


- Problems for future research: other statistics, counts etc.



Prerequisites

Paper


TRASH
and that's Canon.