Extending the Symbolic Method in Enumerative Combinatorics

A Generalization of Pólya's Theorem on Lattice Paths

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Outline

The Symbolic Method: General Principle

Goal: enumerate a countable family $\mathcal{A}$ of combinatorial objects.

- Size function $s\colon \mathcal{A} \to \mathbb{N}_0$ such that, for every n,

$$\mathcal{A}_n = \{\alpha \in \mathcal{A} : s(\alpha) = n\}$$

  is finite.

- Ordinary generating function:

$$A(x) = \sum_{\alpha \in \mathcal{A}} x^{s(\alpha)} = \sum_{n \ge 0} a_n\, x^n \;\in \mathbb{C}[[x]]$$

- With a weight $w\colon \mathcal{A} \to R$ (R a ring), the weighted version:

$$A_w(x) = \sum_{n \ge 0} w(\mathcal{A}_n)\, x^n, \qquad w(\mathcal{A}_n) := \sum_{\alpha \in \mathcal{A}_n} w(\alpha)$$

- Every coefficient $w(\mathcal{A}_n)$ is a finite sum.

The Two Phases of the Method

Formal phase

Combinatorial relations between objects of A become equations (algebraic, differential, functional) between generating functions.

Analytic phase

These equations yield asymptotic formulas for the coefficients.

- The formal phase (the symbolic method) plays a preliminary key role.
- The analytic phase then gives its full power to the method (Flajolet–Sedgewick).

Archetypal Example: Catalan Numbers

$\mathcal{A}$ = proper bracketings, $s(\alpha)$ = number of pairs of brackets.

$$\mathcal{A}_3 = \{\, ()()(),\ ()(()),\ (())(),\ (()()),\ ((()))\,\}$$

$$(a_n)_{n\geq 0} = (1, 1, 2, 5, 14, 42, \ldots) \quad \text{the Catalan numbers}$$

Considering the first pair of brackets yields the algebraic equation

$$A(x) = 1 + x\, A(x)^2$$

Three Efficient Counting Formulas

(a) $a_0 = 1, \quad a_{n+1} = \sum_{j=0}^{n} a_j \cdot a_{n-j}$

(b) $a_n = \frac{1}{n+1}\binom{2n}{n}$

(c) $a_0 = 1, \quad a_{n+1} = \frac{4n+2}{n+2}\, a_n$

Efficient closed formula

An algorithm computing a_n in $O\big((n+1)^c\big)$ bit operations: polynomial time in the combined size of input n and output a_n.

Analytic Phase: Asymptotics

From the equation $xA(x)^2 - A(x) + 1 = 0$ (Flajolet–Sedgewick):

$$a_n = \pi^{-1/2}\, n^{-3/2}\, 4^n\, (1 + o(1)) \qquad (n \to \infty)$$

- A standard, archetypal example: formal phase $\to$ analytic phase.
- Throughout this talk, we extend only the formal phase.

Our Extension: From Finite Sums to Infinite Series

- Drop the assumption "$\mathcal{A}_n$ finite".
- Replace it with: R carries a complete norm (e.g. $R = \mathbb{R}$ or $R = \mathbb{C}$), and

$$\sum_{\alpha \in \mathcal{A}_n} w(\alpha) \quad \text{converges absolutely.}$$

The "semiformal" approach

Coefficients of generating functions may now arise as limits of infinite series.

- Here $R = \mathbb{C}$: we apply this to generalize a classical result — Pólya's theorem on random walks.

Pólya's Theorem (1921)

Let $d, n \in \mathbb{N}$ and $\overline{v} \in \mathbb{Z}^d$. Let $P_d(\overline{v}, n) \in [0, 1]$ be the probability that a walk of length n in the grid graph $\mathbb{Z}^d$, starting at the origin $\overline{0}$, later visits the vertex $\overline{v}$.

Theorem (Pólya, 1921)

For every dimension d and every vertex $\overline{v}$,

$$\lim_{n \to \infty} P_d(\overline{v}, n) \begin{cases} = 1 & \text{if } d \leq 2 \\ \in (0, 1) & \text{if } d \geq 3 \end{cases}$$

Pólya proved it for *any* $\overline{v}$; it is often quoted only for $\overline{v} = \overline{0}$ (recurrence/transience of the walk).

Proof via the Symbolic Method: The Setting

Case $\overline{v} = \overline{0}$. Grid graph $G_d = \langle \mathbb{Z}^d, E_d \rangle$, $2d$-regular and transitive; weight $h\colon E_d \to \{\frac{1}{2d}\}$, extended multiplicatively to walks.

- $\mathcal{A}$: walks starting at $\overline{0}$ that later revisit it $\Rightarrow$ $P_d(\overline{0}, n) = w(\mathcal{A}_n)$
- $\mathcal{B}$: closed walks at $\overline{0}$
- $\mathcal{C} \subseteq \mathcal{B}$: closed walks whose inner vertices are $\neq \overline{0}$
- $\mathcal{D}$: all walks starting at $\overline{0}$

Generating functions $A_w(x), B_w(x), C_w(x), D_w(x) = \sum_{n\geq 0} w(\cdot_n)x^n$, with $D_w(x) = \dfrac{1}{1-x}$.

Key Formal Identities

Formal phase — two identities in $\mathbb{C}[[x]]$:

$$A_w(x) = C_w(x)\, D_w(x) = \frac{C_w(x)}{1-x} \qquad \text{and} \qquad C_w(x) = 1 - \frac{1}{B_w(x)}$$

- Since $w(\mathcal{A}_n) = \sum_{j=0}^{n} w(\mathcal{C}_j)$: the sequence $(w(\mathcal{A}_n))_{n\geq 0}$ is non-decreasing, hence has a limit.
- $B_w(x)$ and $C_w(x)$ have coefficients in $[0,1]$: both converge absolutely for $x \in [0,1)$.

Analytic Phase: A Unified Formula

Abel's theorem (nonnegative coefficients) gives

$$\lim_{n\to\infty} P_d(\overline{0}, n) = C_w(1) = \lim_{x\nearrow 1} C_w(x) = 1 - \frac{1}{B_w(1)}$$

Unified formula

$$\lim_{n\to\infty} P_d(\overline{0}, n) = 1 - \frac{1}{B_w(1)}$$

- $B_w(1) = +\infty \iff d \leq 2; \qquad 1 < B_w(1) < +\infty \iff d \geq 3.$
- This formula unites both cases of Pólya's theorem.

Visiting a Vertex Other Than the Origin

Let $\mathcal{A}_{\overline{v}}$ be the set of walks from $\overline{0}$ visiting $\overline{v}$.

- If $d \leq 2$ (so $B_w(1) = +\infty$):

$$\lim_{n\to\infty} P_d(\overline{v}, n) = 1$$

- If $d \geq 3$:

$$\lim_{n\to\infty} P_d(\overline{v}, n) = \sqrt{1 - \frac{B_{\overline{v},h}(1)}{B_w(1)}}$$

where $\mathcal{B}_{\overline{v}}$ = closed walks at $\overline{0}$ avoiding $\overline{v}$, and $1 \leq B_{\overline{v},h}(1) < B_w(1)$.
(I have not seen this nice square-root closed-form formula in the earlier literature.)

Why Generalize the Graph?

- Idea: replace $\mathbb{Z}^d$ by the countable complete graph

$$K = \left\langle \mathbb{Z}^d,\ \binom{\mathbb{Z}^d}{2} \right\rangle,$$

  with weight 0 outside E_d: nothing essential changes.
- Generalize further by allowing complex weights $w\colon \binom{\mathbb{Z}^d}{2} \to \mathbb{C}$, under a convergence condition.
- For simplicity, replace $\mathbb{Z}^d$ by $\mathbb{N}$. Work on the countable complete graph

$$K_{\mathbb{N}} = \langle \mathbb{N},\ \mathbb{N}_2 \rangle, \qquad \mathbb{N}_2 = \binom{\mathbb{N}}{2}.$$

Walks and Weights on the Complete Graph

- Weight $w\colon \mathbb{N}_2 \to \mathbb{C}$. A walk $\omega = \langle u_0, \dots, u_n \rangle$ satisfies $u_{i-1} \neq u_i$; $s(\omega) = n$.
- Weight of a walk ω: $w(\omega) = \prod_{i=1}^{n} w(\{u_{i-1}, u_i\})$ (and $w(\omega) = 1$ if $n = 0$).
- $\mathcal{D}_n$: walks of length n starting at vertex 1. For $n \geq 1$, $\mathcal{D}_n$ is countably infinite.
- $\mathcal{A}_v$: walks starting at 1 that later visit v.

Definition (light weight)

w is *light* if, for every $n \in \mathbb{N}_0$, the series $\sum_{\omega \in \mathcal{D}_n} w(\omega)$ converges absolutely.

Under this condition $w(\mathcal{A}_{v,n})$ is well defined and $A_{v,h}(x) = \sum_{n \geq 0} w(\mathcal{A}_{v,n}) x^n$.

Four Ways to Sum at the Limit

For $U(x) = \sum_{n\geq 0} u_n x^n$, with $u_n := w(\mathcal{A}_{v,n})$, one has, in decreasing order of precision:

1. Limit $\lim_{n\to\infty} u_n$ of the total weights themselves;
2. Sum $U(1)_2 := \sum_{n\geq 0} u_n$, of an absolutely convergent series;
3. Sum $U(1)_1 := \lim_{n\to\infty} \sum_{j=0}^{n} u_j$, of a conditionally convergent series;
4. Generalized sum $\lim_{x\nearrow 1} U(x)$.

- $U(1)_2$ exists $\Rightarrow$ $U(1)_1$ exists, and $U(1)_1 = U(1)_2$ (the converse fails).
- For nonnegative real weights (the classical, probabilistic case) absolute and conditional convergence coincide. Here, in general, they do not.

Absolute Convergence and Operations on Series

A series $S := \sum_{x \in X} w(x)$ (X countable) converges absolutely if there exists $c > 0$ with $\sum_{x \in Y} |w(x)| \le c$ for every finite $Y \subseteq X$.
(Equivalently, every rearrangement of the terms gives the same sum.)

Three operations (and their propositions)

- **Linear combination**: $\alpha R + \beta S$ converges absolutely, with sum $\alpha r + \beta s$.
- **Grouping** (a partition P of X): same sum as the original series S.
- **Product** $S_1 \times S_2 = \sum_{(x,y)} w_1(x) w_2(y)$: converges absolutely, with sum $s_1 s_2$.

These three propositions replace the finite computations of the classical symbolic method.

Slim Weights and Convex Weights

Slim weight

There is $c(n) > 0$ with $\sum_v |w(\{u,v\})| \le c(n)$ for every vertex u reachable within n steps.
$\Rightarrow$ every slim weight is light.

ξ-convex weight

A light weight h is ξ-convex ($\xi \in \mathbb{C}$) if, for every reachable vertex u,

$$\sum_{v \neq u} w(\{u,v\}) = \xi$$

Case $\xi = 0$: resembles Kirchhoff's node law. If h is ξ-convex and light: $w(\mathcal{D}_n) = \xi^n$, so $D_w(x) = \dfrac{1}{1-\xi x}$.

The Three Abel Theorems

Abel 1 (classical)

If $U(1)_1$ exists, then $\lim_{x \nearrow 1} U(x) = U(1)_1$.

Abel 2

For $U \in \mathbb{R}[[x]]$: if $U(1)_1 = \pm\infty$, then $\lim_{x \nearrow 1} U(x) = U(1)_1$.

Abel 3 (nonnegative coefficients)

For $U \in \mathbb{R}_{\geq 0}[[x]]$: the limit and the sum always exist (possibly $+\infty$) and are equal — a genuine equivalence.

$\Rightarrow$ these convert formal identities into analytic identities at $x = 1$.

Back to the generating functions

With $\mathcal{B}, \mathcal{C}, \mathcal{D}$ defined for walks on $K_{\mathbb{N}}$ starting at 1, exactly as before:

$$A_w(x) = C_w(x)\, D_w(x) \qquad B_w(x) = \frac{1}{1 - C_w(x)}$$

- Proof: cut the walk at the first (resp. every) visit to 1 — a bijection preserving multiplicative weight.
- Unlike the classical method, coefficients are now sums of infinite, absolutely convergent series, not finite sums.

A representative theorem ($U(1)_1$, general weights)

If $A_w(1)_1, B_w(1)_1, C_w(1)_1, D_w(1)_1$ all exist, then $B_w(1)_1 \neq 0$ and

$$A_w(1)_1 = \left(1 - \tfrac{1}{B_w(1)_1}\right) D_w(1)_1$$

Several theorems for walks starting at 1 and revisiting $v = 1$

Depending on the type of convergence and the class of weights, one obtains several theorems for $v = 1$. For example, for $h\colon \mathbb{N}_2 \to \xi\mathbb{R}_{\ge 0}$ ξ-convex and light:

$$\lim_{n\to\infty} \xi^{-n} w(\mathcal{A}_n) = 1 - \frac{1}{(B_w)_{1/\xi}(1)_1} \ \in (0, 1]$$

- Directly generalizes $\lim P_d(\overline{0}, n) = 1 - \frac{1}{B_w(1)}$ (case $d \ge 3$).
- Other versions: absolute convergence $U(1)_2$; a degenerate case $U(1)_1 = \pm\infty$ (limit $= 1$, case $d \le 2$); or fully general complex weights.

The case $v \neq 1$: Transitivity and Square Roots

v-transitivity

There is a bijection $f\colon \mathbb{N} \to \mathbb{N}$ with $f(1) = v$ and $w(e) = w(f[e])$ for every edge e — the analogue of the transitive grid graph.

New generating functions $B_{v,h}$ (avoiding v), $C_{v,h}$, $E_{v,h}$, with

$$A_{v,h}(x) = C_{v,h}(x)D_w(x), \qquad C_{v,h}(x)^2 = 1 - \frac{B_{v,h}(x)}{B_w(x)}$$

A representative theorem (ξ-convex, nonnegative weights)

If $(B_w)_{1/\xi}(1)_1 < +\infty$, then

$$\lim_{n\to\infty} \xi^{-n} w(A_{v,n}) = \sqrt{1 - \frac{(B_{v,h})_{1/\xi}(1)_1}{(B_w)_{1/\xi}(1)_1}} \in (0,1)$$

Monotonicity: $B_{v,h}(x)/B_w(x) \leq 1$ on $(-1,1)$ — generalizing the trivial bound $0 \leq B_{v,h} \leq B_w$.

Summary: Sixteen General Pólya Theorems

Two master formulas, specialized into 16 theorems according to the type of convergence (CC: $U(1)_1$; AC: $U(1)_2$; I: $U(1)_1 = \pm\infty$), ξ-convexity, and whether $v = 1$ or $v \neq 1$:

$$v = 1: \quad A_w \approx \left(1 - \frac{1}{B_w}\right) D_w \qquad\qquad v \neq 1: \quad A_{v,h} \approx D_w \sqrt{1 - \frac{B_{v,h}}{B_w}}$$

$v = 1$ — 7 theorems (§3)

- CC $A = \left(1 - \frac{1}{B}\right) D$, or $\lim \xi^{-n} w(\mathcal{A}_n) = 1 - \frac{1}{(B)_{1/\xi}}$
- AC same conclusion, under absolute convergence
- I $B_w(1)_1 = +\infty \;\Rightarrow$ limit $= +\infty$ or $= 1$

$v \neq 1$ — 9 theorems (§4)

- CC $A_v = D\sqrt{1 - \frac{B_v}{B}}$
- AC same conclusion, under absolute convergence
- I limit $= +\infty$ or $\in \{-1, 1\}$

The passage $v = 1 \to v \neq 1$ systematically introduces a square root.

Conclusion and Perspectives

- Pólya's theorem becomes a result in enumerative combinatorics: no Markov chains needed.
- The only new ingredient: the formal extension — finite sums $\to$ absolutely convergent series.

Related and forthcoming work

- A general "semiformal" theory of formal power series (Paper II, in preparation).
- Weights in $\mathbb{C}[[x_1, \ldots, x_k]]$, with a non-Archimedean norm.
- A semiformal approach in number theory: two proofs of the transcendence of e.
- Open question: several vertices v visited simultaneously.

Thank you for your attention!