Lattice paths and the toric g-vector of nestohedra

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The main illustration

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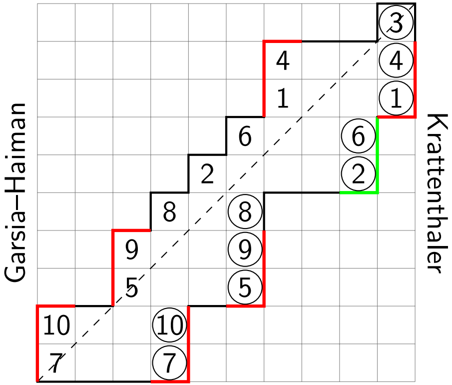

1. Toric g-vectors
   - The toric g-vector of a simple polytope
   - The toric g-vector of the associahedron

2. Generalizations and variants
   - Variants
   - Parking trees

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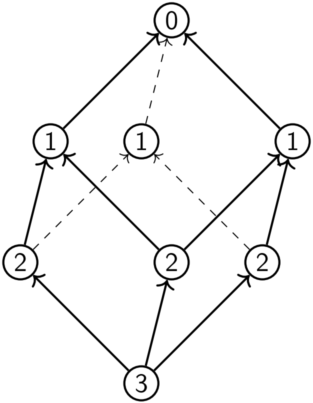

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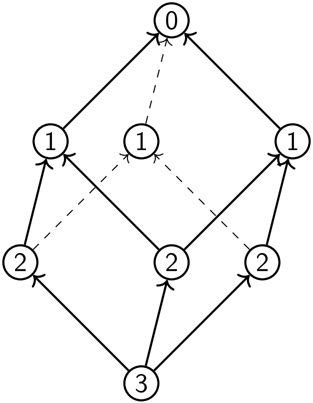

$h_0 = 1,\ h_1 = 3,\ h_2 = 3,\ h_3 = 1.$

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$$f([\widehat{0},\widehat{1}),x) = \sum_{p \in [\widehat{0},\widehat{1})} g([0,p),x)(x-1)^{\rho(\widehat{1})-1-\rho(t)} \quad \text{and}$$

$$g([\widehat{0},\widehat{1}),x) = \sum_{i=0}^{\lfloor(\rho(\widehat{1})-1)/2\rfloor} ([x^i]f([\widehat{0},\widehat{1}),x) - [x^{i-1}]f([\widehat{0},\widehat{1}),x))x^i$$

subject to the initial conditions $f(\emptyset,x) = g(\emptyset,x) = 1$.

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3. Stanley defined toric h- and a toric g-polynomials for (lower) Eulerian posets, using an intertwined recurrence. (It is complicated and it is not self-dual. The toric h-vector is same as the usual h-vector for simplicial polytopes.)
4. Stanley asked for a formula expressing the toric g-vector of the (face lattice) of the permutahedron. (Now we have an answer.)

Lemma (Hetyei)

The toric g-polynomial of an n-dimensional simple polytope P is given by

$$g(P, x) = h_0 + \sum_{i=1}^{\lfloor n/2 \rfloor} (h_i - h_{i-1}) \cdot C(n, i, x).$$

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Also note that $h_i - h_{i-1} = \sum_{j=0}^{i} C(n - 2j, i - j) \cdot \gamma_j$ holds for the *gamma vector*.

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Lemma

The coefficient of x^k in $\sum_{i=j}^{\lfloor n/2 \rfloor} C(n-2j, i-j) \cdot C(n,i,x)$ is the number of Dyck paths of semilength $n-j$ with k peaks whose first coordinate is at most $n-1$.

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where $(\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor})$ is the γ-vector of the simple polytope P.

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In our work we used the results of Postnikov–Reiner–Williams. We also answered Stanley's original question to understand the toric g-vector of the permutahedron. In this talk we focus on the associahedron whose gamma vector is

$$\gamma_j(\mathcal{A}_n) = C_j \cdot \binom{n}{2j}.$$

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The coefficient of x^k in the toric g-polynomial of the n-dimensional associahedron is the number of 123-avoiding parking functions on the set $[n]$ having exactly k ascents.

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The number of all 123-avoiding parking functions was first computed by Qiu and Remmel.

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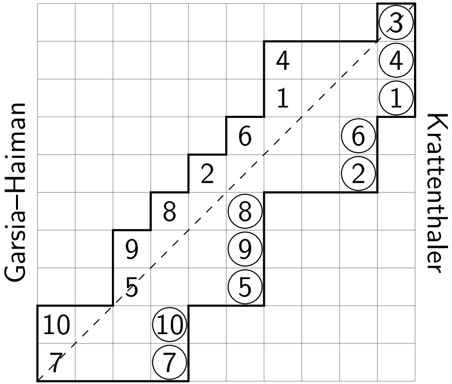

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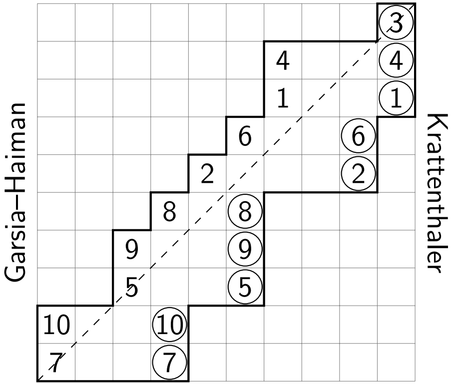

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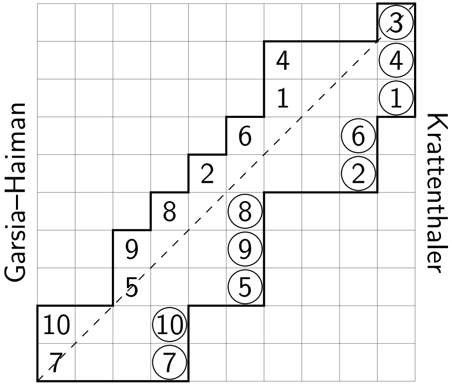

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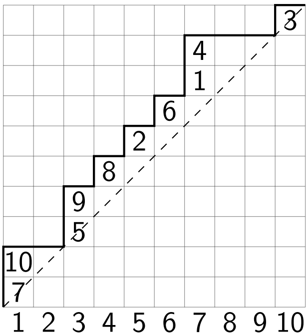

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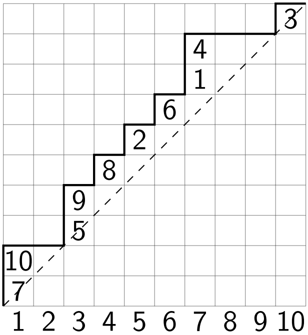

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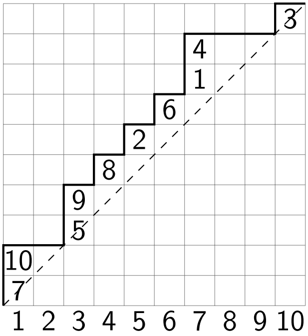

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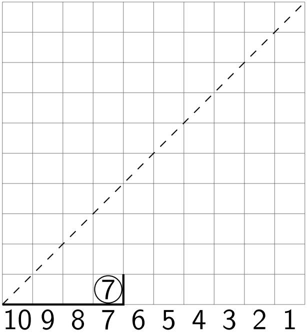

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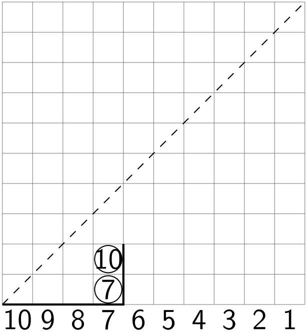

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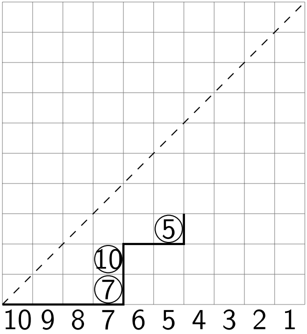

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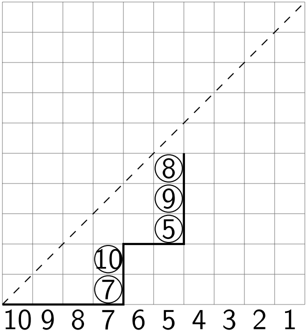

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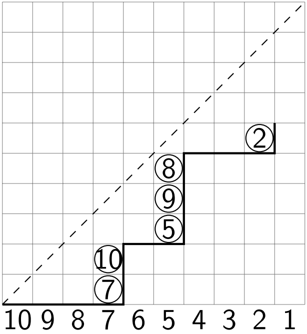

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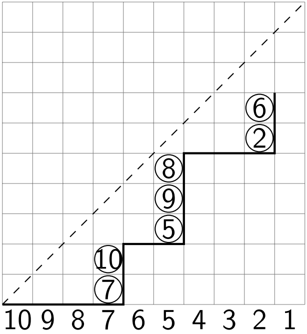

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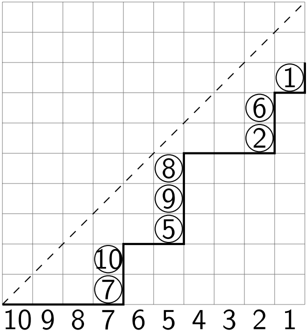

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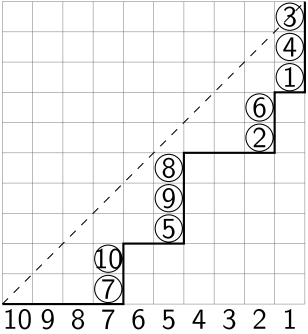

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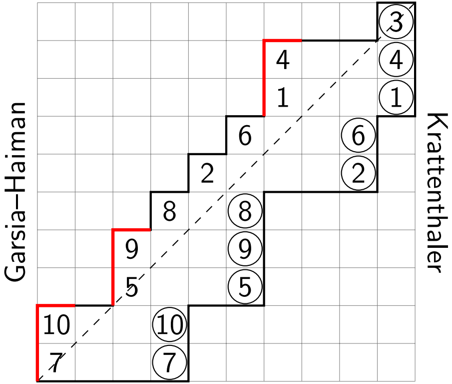

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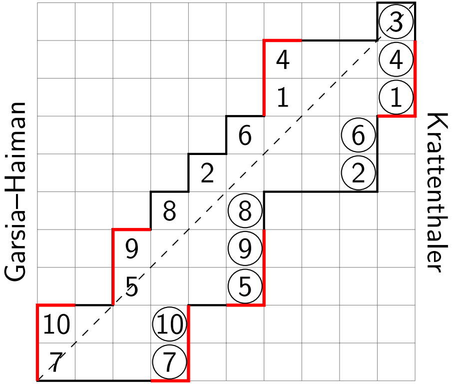

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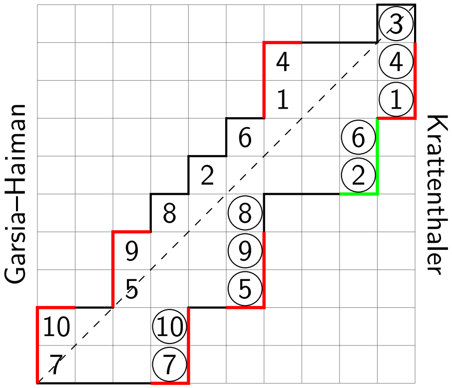

Counting compatible Dyck words

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Definition

Let A and B be two sparse subsets of the set $[n-1]$ and let w be a labeled Dyck word of semilength n. We say that the word w is (A, B)-compatible if

(a) for each $a \in A$ the word $U_a U_{a+1} D$ is a factor in w, and

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The proof is an explicit recursive bijection.

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Corollary

Let A and B be sparse subsets of $[n-1]$. Then the number of 123-avoiding permutations π in $\mathfrak{S}_n$ such that $B \subseteq \mathrm{Asc}(\pi)$ and $A \subseteq \mathrm{Asc}(\pi^{-1})$ is given by the Catalan number $C_{n-|A|-|B|}$.

The toric contribution of the γ-vector entries

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Proposition

Let B be a sparse subset of the set $[n-1]$. The coefficient of x^k in the toric g-contribution polynomial $g_{n,|B|}(x)$ is given by:

(a) *the number of all $(\emptyset, B)$-compatible Dyck words of semilength n that contain exactly k copies of the word UUD as a factor.*

(b) *the number of 123-avoiding permutations $\pi \in \mathfrak{S}_n$ such that $B \subseteq \mathrm{Asc}(\pi)$ and the inverse permutation π^{-1} has exactly k ascents.*

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The kth entry in the toric g-vector of the n-dimensional cube is the number of 123*-avoiding permutations of* $[n]$ *having exactly* k *ascents.*

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The kth entry in the toric g-vector of the n-dimensional cube is the number of 123-avoiding permutations of $[n]$ having exactly k ascents.

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The coefficient of x^k in the toric g-polynomial of the n-dimensional cyclohedron is the number of 123-avoiding functions $f : [n] \longrightarrow [n]$ having exactly k ascents.

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Parking trees

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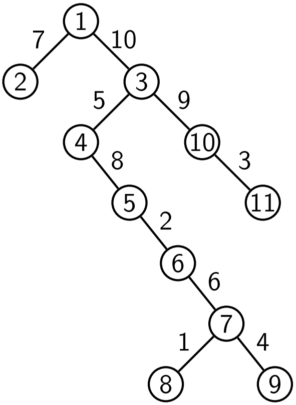

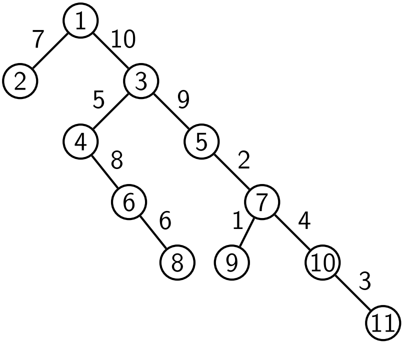

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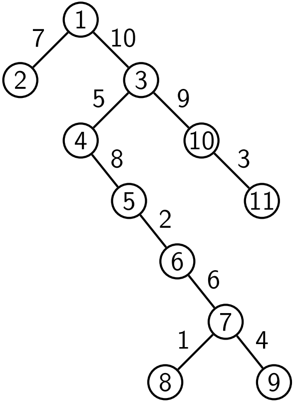

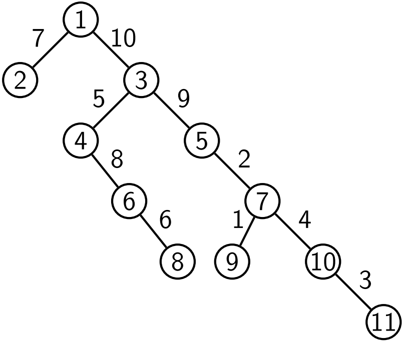

Depth-first search and breadth-first search parking trees of the same parking function.

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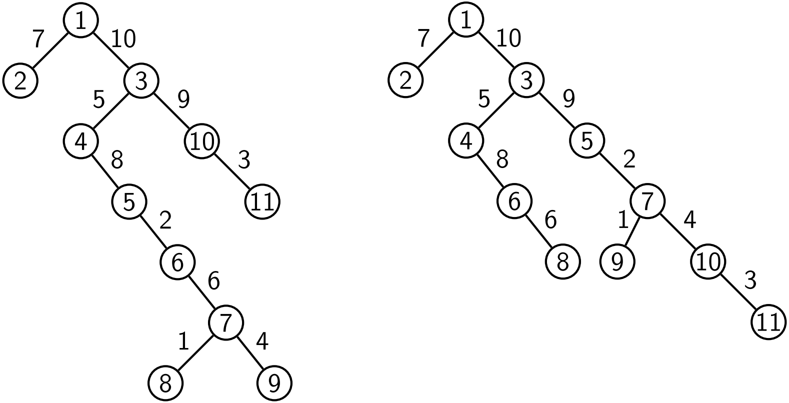

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The toric g-vector of the permutahedron

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Thank you!

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https://ecajournal.haifa.ac.il/Conference/ICECA2026.html