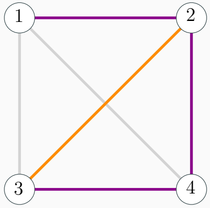
Tree walks

Eva-Maria Hainzl

LIX, École polytechnique, CNRS, Institut Polytechnique de Paris
LPC, Vienna 2026

Graphs and adjacency matrices

Graph $G = (V, E)$



Adjacency matrix $A(G)$

	1	2	3	4
1	0	1	0	0
2	1	0	1	1
3	0	1	0	1
4	0	1	1	0

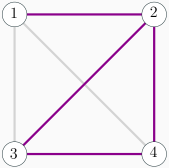
$$V = \{1, 2, 3, 4\}, \quad E = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$$

The spectrum of graphs

Graph spectrum

The spectrum of a graph is the set of eigenvalues of its adjacency matrix.

$$sp(G) = \{\lambda \in \mathbb{R} \mid \det(\lambda I - A(G)) = 0\}$$



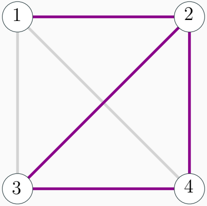
$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\det(\lambda I - A(G)) = (\lambda + 1)(\lambda^3 - \lambda^2 - 3\lambda + 1) = 0$$

RANDOM graphs and adjacency matrices

Random graph

$G(n, p) = (V, E)$



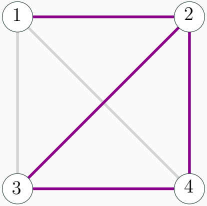
$$V = \{1, 2, 3, 4\},$$

$$\mathbb{P}\Big(E = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}\Big) = p^4(1 - p)^{6-4}$$

RANDOM graphs and adjacency matrices

Random graph

$G(n, p) = (V, E)$



Random adjacency matrix

$A(G)$ with entries $\sim Ber(p)$

	1	2	3	4
1	0	1	0	0
2	1	0	1	1
3	0	1	0	1
4	0	1	1	0

$$V = \{1, 2, 3, 4\},$$

$$\mathbb{P}\Big(E = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}\Big) = p^4(1-p)^{6-4}$$

The spectrum of random graphs

The empirical spectral density

For a random graph $G(n,p)$, the empirical spectral density is

$$\mu_n(z) = \frac{1}{n} \sum_{\lambda \in sp(1/\sqrt{pn}\, A(G))} \mathbf{1}_{\{\lambda\}}(z)$$

(= essentially a random histogram)

Given p, how do we expect μ_n to behave for large n?

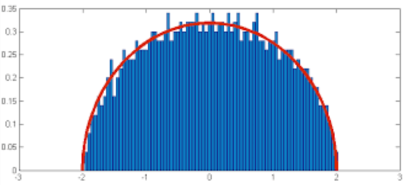
Wigner's law

(A generalization of) Wigner's law

If $np \to \infty$, then the empirical spectral density μ_n of $G(n,p)$ converges to the semicircle distribution

$$\mu_n(z) \to \left(\frac{1}{2\pi}\sqrt{4-z^2}\right)\mathbf{1}_{|z|<2}$$

Ex.: Histogram of spectrum of $G(n, 1/2)$.

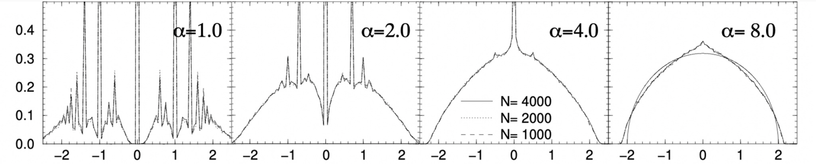


Random graphs

What if $np \to \alpha$?

- There exists a limit distribution μ^α (Zacharevich, 2006)

Ex.: Histograms of spectra of $G(N,p)$ with $p = \alpha/N$.



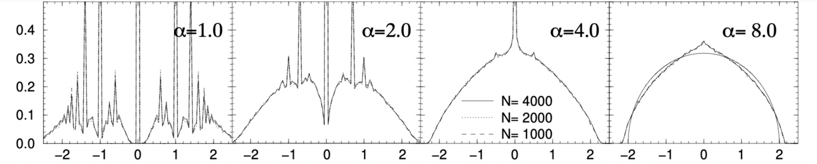
Plots by Bauer and Golinelli, 2001

Random graphs

What if $np \to \alpha$?

- There exists a limit distribution μ^α (Zacharevich, 2006)
- Moments can be computed by counting tree walks (Bauer, Golinelli, 2001)

Ex.: Histograms of spectra of $G(N, p)$ with $p = \alpha/N$.



Plots by Bauer and Golinelli, 2001

The moments of μ_n^α

Computing the moments...

$$m_k(\mu_n^\alpha) = \sum_G \mathbb{P}\left[G\left(n, {}^{\alpha}\!/_{n}\right) = G\right] \cdot \frac{1}{n} \sum_{\lambda \in sp(1/\sqrt{\alpha}\, A(G))} \lambda^k$$

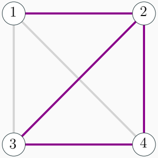
The moments of μ_n^α

Computing the moments...

$$m_k(\mu_n^\alpha) = \sum_G \mathbb{P}\left[G(n, {}^{\alpha}/_{n}) = G\right] \cdot \frac{1}{n} \sum_{\lambda \in sp(1/\sqrt{\alpha}\, A(G))} \lambda^k$$

$$\sum_{i=1}^{n} \lambda_i^k = tr\left(A^k\right)$$

where $(A^k)_{i,j} =$ number of walks from i to j of length k.



$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad A(G)^3 = \begin{pmatrix} 0 & 3 & 1 & 1 \\ 3 & 2 & 4 & 4 \\ 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

The moments of μ_n^α

Computing the moments... (Enriquez, Ménard, 2016)

$$m_k(\mu_n^\alpha) = \sum_G \mathbb{P}\left[G\left(n, {}^{\alpha}/_{n}\right) = G\right] \cdot \frac{1}{n} \sum_{\lambda \in sp(1/\sqrt{\alpha}\, A(G))} \lambda^k$$

For $k \geq 1$, as $n \to \infty$,

$$m_{2k-1}(\mu_n^\alpha) \sim 0$$

The moments of μ_n^α

Computing the moments... (Enriquez, Ménard, 2016)

$$m_k(\mu_n^\alpha) = \sum_G \mathbb{P}\left[G\left(n, {}^{\alpha}/_{n}\right) = G\right] \cdot \frac{1}{n} \sum_{\lambda \in sp(1/\sqrt{\alpha}\, A(G))} \lambda^k$$

For $k \geq 1$, as $n \to \infty$,

$$m_{2k-1}(\mu_n^\alpha) \sim 0$$

and

$$m_{2k}(\mu_n^\alpha) \sim \sum_{i=1}^{\min\{n,k\}} \frac{w_{k,i+1}}{(i+1)!} \alpha^{i-k},$$

where $w_{k,i+1}$ = number of walks of length $2k$ spanning a tree with $i+1$ labeled vertices

The moments of μ_n^α

Computing the moments... (Enriquez, Ménard, 2016)

$$m_k(\mu_n^\alpha) = \sum_G \mathbb{P}\left[G\left(n, {}^{\alpha}/_{n}\right) = G\right] \cdot \frac{1}{n} \sum_{\lambda \in sp(1/\sqrt{\alpha}\, A(G))} \lambda^k$$

For $k \geq 1$, as $n \to \infty$,

$$m_{2k-1}(\mu_n^\alpha) \sim 0$$

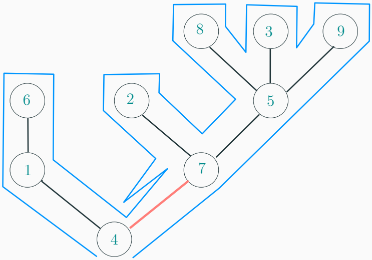
and

$$m_{2k}(\mu_n^\alpha) \sim \sum_{i=1}^{\min\{n,k\}} \frac{w_{k,i+1}}{(i+1)!} \alpha^{i-k},$$

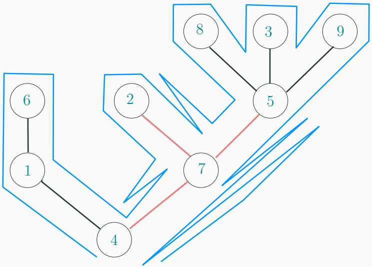
where $w_{k,i+1}$ = number of walks of length $2k$ spanning a tree with $i+1$ labeled vertices and $k-i$ =: excess of the walk

Excess of tree walks

Excess = # steps/2 - # edges



Ex: Excess 1 walk



Ex: Excess 5 walk

Generating functions of μ^α

Associated generating functions as $n \to \infty$

$$M_{\alpha,n}(z) = \sum_{k=1}^{\infty} \sum_{i=1}^{\min\{n,k\}} \frac{w_{k,i+1}}{(i+1)!} \alpha^{i-k} z^{2k}$$

$w_{k,i+1} =$ # walks of length $2k$ spanning tree with $i+1$ vertices

Generating functions of μ^α

Associated generating functions as $n \to \infty$

$$M_{\alpha,n}(z) = \sum_{k=1}^{\infty} \sum_{i=1}^{\min\{n,k\}} \frac{w_{k,i+1}}{(i+1)!} \alpha^{i-k} z^{2k}$$

$$= \sum_{k=1}^{\infty} \frac{w_{k,k+1}}{(k+1)!} z^{2k} + \frac{1}{\alpha} \sum_{k=1}^{\infty} \frac{w_{k,k}}{k!} z^{2k} + \frac{1}{\alpha^2} \sum_{k=1}^{\infty} \frac{w_{k,k-2}}{(k-2)!} z^{2k} + \dots$$

$w_{k,i+1}$ = # walks of length $2k$ spanning tree with $i+1$ vertices

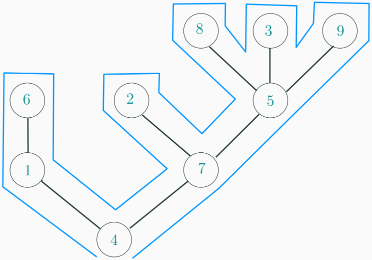
$$\implies W_i(z) = \sum_{k=1}^{\infty} \frac{w_{k,k+1-i}}{(k+1-i)!} z^k$$

GF of tree walks with excess i

What is $W_0(z)$?

What is # walks of length $2k$ spanning tree with $k+1$ vertices?

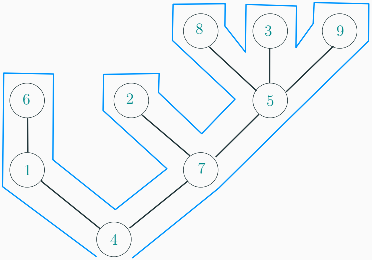
$\Longrightarrow$ Walk defines a plane embedding



What is $W_0(z)$?

What is # walks of length $2k$ spanning tree with $k+1$ vertices?

$\Longrightarrow$ Walk defines a plane embedding

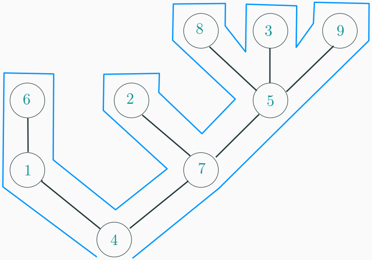


The Catalan numbers $\frac{1}{k+1}\binom{2k}{k}$, so $W_0(z) = \frac{1-\sqrt{1-4z}}{2z}$!

What is $W_0(z)$?

What is # walks of length $2k$ spanning tree with $k+1$ vertices?

$\Longrightarrow$ Walk defines a plane embedding



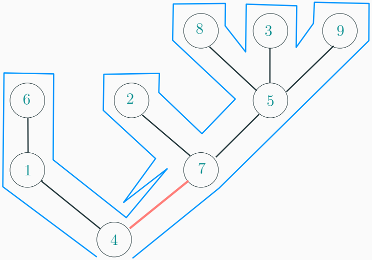
The Catalan numbers $\frac{1}{k+1}\binom{2k}{k}$, so $W_0(z) = \frac{1-\sqrt{1-4z}}{2z}$!

They are moments of the semicircle law!

What is $W_1(z)$?

Enriquez, Ménard, 2016

$$W_1(z) = \frac{z^2 C(z)^5}{1 - zC(z)^2} \quad \text{with} \quad [z^k]W_1(z) = \binom{2k}{k-2}, \quad k \geq 2$$

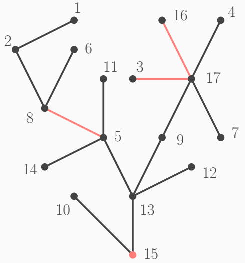


What is $W_i(z)$?

de Panafieu, H., 2024

$$W_i(z) = C(z) \sum_{j=0}^{2i-2} \frac{R_{i,j}\left(zC(z)^2\right)}{\left(1 - zC(z)^2\right)^{j+1}}$$

where $C(z)$ is the Catalan generating function and $R_{i,j}(x)$ are polynomials with non-neg. coefficients of degree $2i + j$.

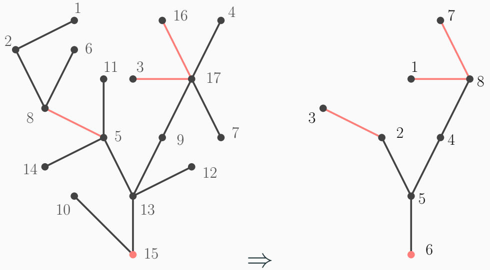


What is $W_i(z)$?

de Panafieu, H., 2024

$$W_i(z) = C(z) \sum_{j=0}^{2i-2} \frac{R_{i,j}\left(zC(z)^2\right)}{\left(1 - zC(z)^2\right)^{j+1}}$$

where $C(z)$ is the Catalan generating function and $R_{i,j}(x)$ are polynomials with non-neg. coefficients of degree $2i + j$.

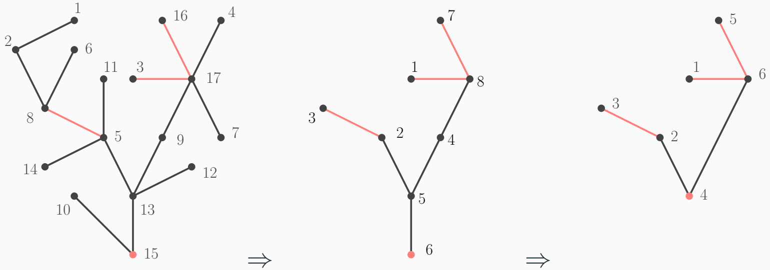


What is $W_i(z)$?

de Panafieu, H., 2024

$$W_i(z) = C(z) \sum_{j=0}^{2i-2} \frac{R_{i,j}\left(zC(z)^2\right)}{\left(1 - zC(z)^2\right)^{j+1}}$$

where $C(z)$ is the Catalan generating function and $R_{i,j}(x)$ are polynomials with non-neg. coefficients of degree $2i + j$.



Asymptotics of the moments

de Panafieu, H., 2024

$$W_i(z) = C(z) \sum_{j=0}^{2i-2} \frac{R_{i,j}\left(zC(z)^2\right)}{\left(1 - zC(z)^2\right)^{j+1}}$$

where $C(z)$ is the Catalan generating function and $R_{i,j}(x)$ are polynomials with non-neg. coefficients of degree $2i + j$.

✓ Asymptotics for fixed excess i.

Asymptotics of the moments

de Panafieu, H., 2024

$$W_i(z) = C(z) \sum_{j=0}^{2i-2} \frac{R_{i,j}\left(zC(z)^2\right)}{\left(1 - zC(z)^2\right)^{j+1}}$$

where $C(z)$ is the Catalan generating function and $R_{i,j}(x)$ are polynomials with non-neg. coefficients of degree $2i + j$.

✓ Asymptotics for fixed excess i.

? What about the asymptotics of

$$[z^n] \left(W_0(z^2) + \frac{1}{\alpha} W_1(z^2) + \frac{1}{\alpha^2} W_2(z^2) + \dots \right)$$

Asymptotics of ℓ-tours

Theorem (H., 2025) [Conjectured by Gunnels, 2021]

Let $\ell \geq 1$ and $c_m^{(\ell)}$ be the number of tree walks spanning a tree with $m + 1$ vertices, where each edge is crossed exactly ℓ-times.

Asymptotics of ℓ-tours

Theorem (H., 2025) [Conjectured by Gunnels, 2021]

Let $\ell \geq 1$ and $c_m^{(\ell)}$ be the number of tree walks spanning a tree with $m+1$ vertices, where each edge is crossed exactly ℓ-times. Then

$$c_m^{(1)} \sim \frac{1}{\sqrt{\pi m^3}} 4^m, \qquad c_m^{(2)} \sim 2\sqrt{\frac{e^3}{\pi m}}\, 2^m m!$$

and

$$c_m^{(\ell)} \sim 2\sqrt{\frac{\ell}{(2\pi m)^{\ell-1}}} \left(\frac{\ell^\ell}{\ell!}\right)^n (n!)^{\ell-1}, \quad \ell \geq 3$$

as $n \to \infty$.

(Gunnels called c_m^ℓ the *hypergraph Catalan numbers*.)

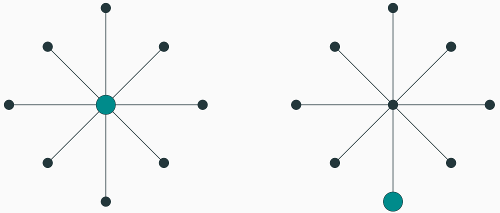
Asymptotics of hypergraph Catalan numbers

Maximize number of mixings

Consider a tree with one vertex with degree m (the rest are leaves):

- number of plane trees: 2
- number of ways to mix:

$$\frac{1}{m!}\binom{\ell m}{\ell, \ell, \ldots, \ell} \sim \sqrt{\frac{\ell}{(2\pi m)^{\ell-1}}}\left(\frac{\ell^\ell}{\ell!}\right)^m (m!)^{\ell-1}$$



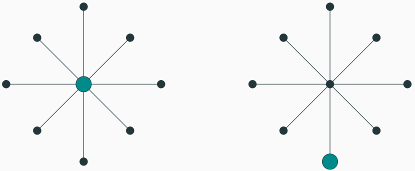
What about tree walks by excess?

Tree walks of length $2n$ with excess k

Consider a tree with one vertex with degree n:

- number of plane trees: 2
- number of ways to mix steps: the Stirling number of 2nd kind

$$S(n, n-k) = \left\{ \begin{matrix} n \\ n-k \end{matrix} \right\}$$



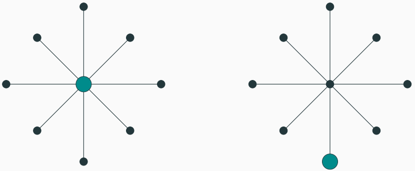
What about tree walks by excess?

Tree walks of length $2n$ with excess k

Consider a tree with one vertex with degree n:

- number of plane trees: 2
- number of ways to mix steps: the Stirling number of 2nd kind

$$S(n, n-k) = \left\{ {n \atop n-k} \right\}$$



Unfortunately more trees contribute, but max. if $n - k \sim \frac{n}{\log n}$...

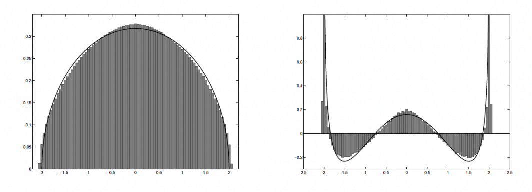
More open questions

Theorem (Enriquez, Ménard, 2016)

$$m_{2k}(\mu^\alpha) = m_{2k}\left(\sigma + \frac{1}{\alpha}\sigma_1\right) + o\left(\frac{1}{\alpha}\right)$$

where σ is the semicircle law and σ_1 a measure with density

$$\frac{1}{2\pi}\frac{z^4 - 4z^2 + 2}{\sqrt{4-z^2}}\mathbf{1}_{|z|<2}$$



$\alpha = 20, n = 10000, N = 100$

An alternative scaling of μ_n^α

The moments of μ_n^α

$$m_k(\mu_n^{\alpha}) = \sum_G \mathbb{P}\left[G\left(n, {}^{\alpha}/_{n}\right) = G\right] \cdot \frac{1}{n} \sum_{\lambda \in sp(1/\sqrt{\alpha}\, A(G))} \lambda^k$$

For $k \geq 1$, as $n \to \infty$,

$$m_{2k}(\mu_n^\alpha) \sim \sum_{i=1}^{k} \frac{w_{k,i+1}}{(i+1)!} \frac{\alpha^i}{\alpha^k},$$

where $w_{k,i+1}$ = number of walks of length $2k$ spanning a tree with $i+1$ labeled vertices

An alternative scaling of μ_n^α

The moments of $\mu_n^{\alpha f(\alpha^{-1})}$

$$m_k(\mu_n^{\alpha f(\alpha^{-1})}) = \sum_G \mathbb{P}\left[G(n, \alpha/n) = G\right] \cdot \frac{1}{n} \sum_{\lambda \in sp(1/\sqrt{\alpha f(\alpha^{-1})}\, A(G))} \lambda^k,$$

where $f(x)$ some function/power series.

For $k \geq 1$, as $n \to \infty$,

$$m_{2k}(\mu_n^\alpha) \sim \sum_{i=1}^{k} \frac{w_{k,i+1}}{(i+1)!} \frac{\alpha^i}{(\alpha f(\alpha^{-1}))^k}$$

An alternative scaling of μ_n^α

The moments of $\mu_n^{\alpha f(\alpha^{-1})}$

$$m_k(\mu_n^{\alpha f(\alpha^{-1})}) = \sum_G \mathbb{P}\left[G\left(n, {}^{\alpha}/_{n}\right) = G\right] \cdot \frac{1}{n} \sum_{\lambda \in sp(1/\sqrt{\alpha f(\alpha^{-1})}\, A(G))} \lambda^k,$$

where $f(x)$ some function/power series.

For $k \geq 1$, as $n \to \infty$,

$$m_{2k}(\mu_n^\alpha) \sim \sum_{i=1}^{k} \frac{w_{k,i+1}}{(i+1)!} \frac{\alpha^i}{(\alpha f(\alpha^{-1}))^k}$$

A new expansion in α

$$M_{\alpha f(\alpha^{-1})}(z) \sim V_0(z) + \frac{1}{\alpha} V_1(z) + \frac{1}{\alpha^2} V_2(z) + \ldots$$

What is $V_i(z)$?

de Panafieu, H., 2024

If $f(x) = 1 + x + x^2 + 4x^3 + 33x^4 + 386x^5$ then

$$V_i(z) = C(z) Q_i\left(zC(z)^2\right), \quad i = 0, 1, 2, \ldots, 5,$$

where

$$\begin{aligned}
&Q_0(x) = 1, \qquad Q_1(x) = -x, \qquad Q_2(x) = -2x^3, \\
&Q_3(x) = -\left(11x^5 + x^4 - 2x^3 + 2x^2 + 3x\right), \\
&Q_4(x) = -\left(90x^7 + 27x^6 - 19x^5 + 17x^4 + 23x^3 + 20x^2 + 26x\right), \\
&Q_5(x) = -\Big(931x^9 + 529x^8 - 163x^7 + 166x^6 + 301x^5 + 239x^4 \\
&\qquad\qquad + 249x^3 + 266x^2 + 324x\Big).
\end{aligned}$$

What is $V_i(z)$?

Conjecture (de Panafieu, H., 2024)

There exists a unique power series $F(x)$ with non-negative coefficients such that for

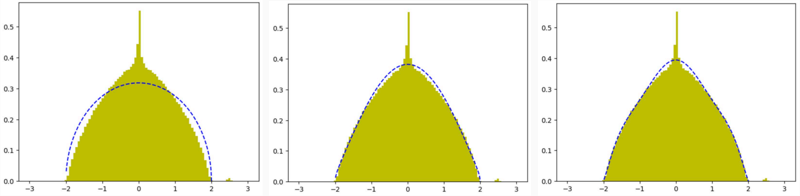
$$\tilde{W}_i(z) = [\alpha^{-i}]M_{\alpha F(\alpha^{-1})}(z)$$

there exists a polynomial $Q_i(x)$ such that

$$\tilde{W}_i(z) = C(z)Q_i\left(zC(z)^2\right).$$

Densities from $V_i(z)$, $i = 0, 1, 2$

Ex: Histograms (100 bins) of eigenvalues of $N = 100$ random adjacency matrices of $G(n, 5/n)$, $n = 1000$ compared to the given densities.

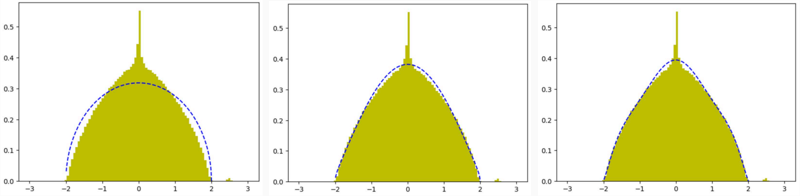


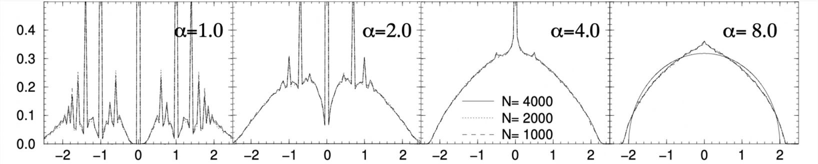
$f_0(z)$ $\qquad$ $f_0(z) + \frac{1}{5} f_1(z)$ $\qquad$ $f_0(z) + \frac{1}{5} f_1(z) + \frac{1}{25} f_2(z)$

$$f_0(z) = \frac{\sqrt{4 - z^2}}{2\pi} \mathbf{1}_{(-2,2)}(z), \; f_1(z) = \frac{(1 - z^2)\sqrt{4 - z^2}}{2\pi} \mathbf{1}_{(-2,2)}(z), \; f_2(z) = \frac{(1 - 6z^2 + 5z^4 - z^6)\sqrt{4 - z^2}}{2\pi} \mathbf{1}_{(-2,2)}(z).$$

Densities from $V_i(z)$, $i = 0, 1, 2$

Ex: Histograms (100 bins) of eigenvalues of $N = 100$ random adjacency matrices of $G(n, 5/n)$, $n = 1000$ compared to the given densities.





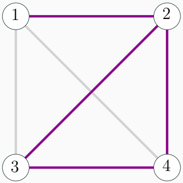
Open question

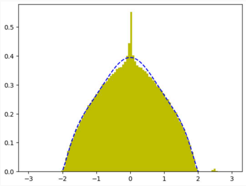
Lots of research questions remain...

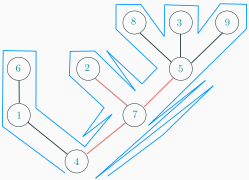
- What is the combinatorial meaning of the scaling by

$$f(x) = 1 + x + x^2 + 4x^3 + 33x^4 + 386x^5?$$

- What are the asymptotics of $m_{2n}(\mu_n^\alpha)$ (and $m_{2n}(\mu_n^\alpha)$)?
- Other specializations of tree walks (e.g. k-tours)







Thank you for the attention :)