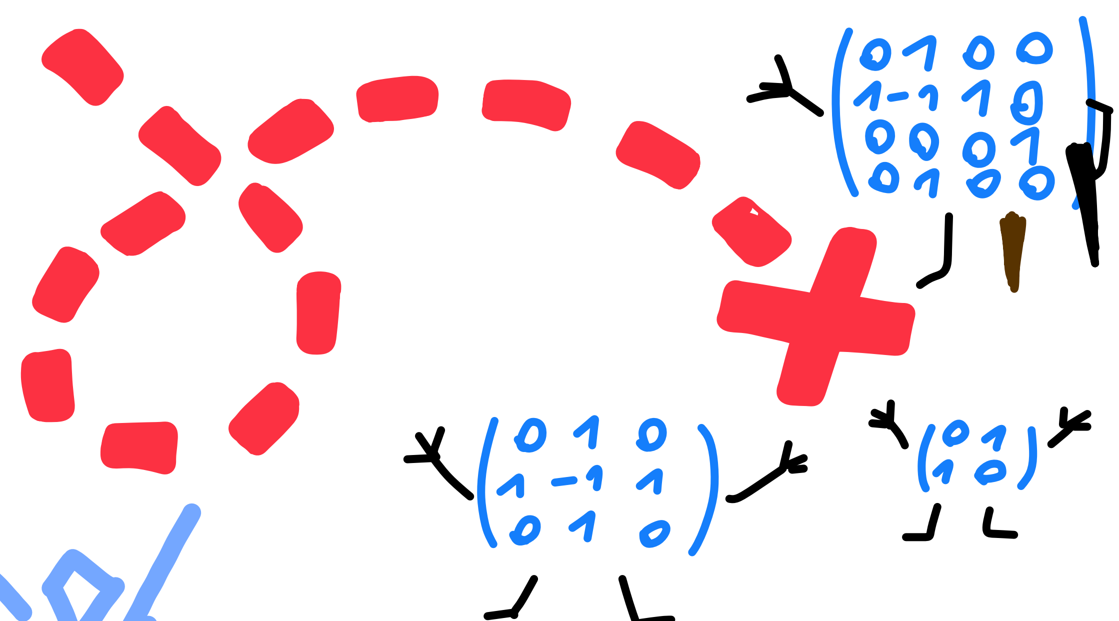
2  identities for inhomogeneous Hall-Littlewood functions



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Moritz Gangl
(Joint work with Ilse Fischer)

"The" Littlewood identity

$$\sum_{\lambda_1 \geq \dots \geq \lambda_n \geq 0} \underbrace{S_{(\lambda_1,\dots,\lambda_n)}}_{\text{Schur-function}}(x_1,\dots,x_n) = \underbrace{\prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1\leq i<j\leq n} \frac{1}{1-x_i x_j}}_{\text{"Littlewood-kernel"}}$$

Littlewood identities

Thm: [Fischer 2024]

For any $n \geq 0$:

$$\sum_{0 \leq k_1 < k_2 < \ldots < k_n} R_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;1,1,1,w) = \prod_{i=1}^{n} \frac{x_i^{-1}+(1+w)+x_i}{1-x_i} \prod_{1 \leq i < j \leq n} \frac{1+x_i+x_j+wx_ix_j}{1-x_ix_j}$$

Thm: [Fischer, Höngesberg 2025]

Littlewood-kernel on RHS!

$$\sum_{0 \leq k_1 < \ldots < k_n} R^*_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;1,1,w) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1 \leq i < j \leq n} \frac{1}{1-x_ix_j} Z_{DSASM}(x_1,\ldots,x_n)$$

Littlewood identities

Thm: [Fischer 2024]

Robbins polynomial

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modified Robbins polynomial

Littlewood-kernel on RHS!

New Littlewood identities

Thm 1: [Fischer, G. 2025+]

For any $n, p \geq 0$ & $s_j = s_p \ \forall j \geq p$:

$$\sum_{\lambda_1 \geq \dots \geq \lambda_n \geq 0} \prod_{r \geq 0} \frac{(-s_r;q)_{m_r(\lambda)}}{(q;q)_{m_r(\lambda)}} F_\lambda(u_1, \dots, u_n | s_0, s_1, \dots)$$

$$= \prod_{i=1}^{n} \frac{1}{1-u_i} \prod_{1 \leq i < j \leq n} \frac{1 - q u_i u_j}{1 - u_i u_j}$$

Littlewood-kernel appears again!

New Littlewood identities

Thm 1: [Fischer, G. 2025+]

For any $n, p \geq 0$ & $s_j = s_p \ \forall j \geq p$:

fully inhomogeneous spin Hall-Littlewood rational symmetric functions

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Thm 2: [Fischer, G. 2025+]

For any $n, p \geq 0$ & $s_j = s_p \; \forall j \geq p$:

$$\sum_{\lambda_1 \geq \ldots \geq \lambda_n \geq 0} \frac{(-\gamma q^{1/2}; q^{1/2})_{m_0(\lambda)} (-\gamma^{-1} s_0; q^{1/2})_{m_0(\lambda)}}{(q;q)_{m_0(\lambda)}} \times \prod_{r \geq 1} \frac{(-s_r; q^{1/2})_{m_r(\lambda)}}{(q^{1/2}; q^{1/2})_{m_r(\lambda)}} F_\lambda(u_1, \ldots, u_n \mid s_0, s_1, \ldots)$$

$$= \prod_{i=1}^{n} \frac{1+q^{1/2}}{1-u_i} \prod_{1 \leq i < j \leq n} \frac{1-q u_i u_j}{u_i - u_j} \operatorname{Pf}(M^{(\gamma)})$$

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where

$$M^{(\gamma)} = \left(\begin{cases} 1 + (\gamma - 1) \dfrac{(q^{1/2} - \gamma^{-1} s_0)(1-u_j)}{(1+q^{1/2})(1-s_0 u_j)}, & i = 0, \\[2ex] \dfrac{(u_i - u_j)\Big(\big((1+q)(1-q^{1/2} u_i u_j) + (u_i + u_j)(q^{1/2} - q)\big) + (\gamma - 1) f(u_i, u_j)\Big)}{(1+q^{1/2})(1-u_i u_j)(1-q u_i u_j)}, & i \geq 1 \end{cases} \right)_{\chi_{\text{even}}(n) \leq i < j \leq n}$$

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and $f(u,v) =$

$$\frac{(q^{1/2} - \gamma^{-1} s_0)(1 - uv)}{(1 + q^{1/2})(1 - s_0 u)(1 - s_0 v)} \Big((1 + q^{1/2})(1 + \gamma^{-1} s_0)(1 + q \gamma u v) + (1 + \gamma)(q - \gamma^{-1} s_0)(1 - q^{1/2} u v) - (1+\gamma) q^{1/2} (q^{1/2} + \gamma^{-1} s_0)(u + v) \Big)$$

Ok cool... but why is that interesting?

1) All Littlewood identities for F_λ in literature (see e.g.: Chen & Ding '21, Gavrilova '21, ...) sum over partitions with λ' even!
So far no identities summed over all partitions with their methods. We use different methods to prove this!

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$\rightsquigarrow$ Cauchy identity, Littlewood identities for R, R^*!
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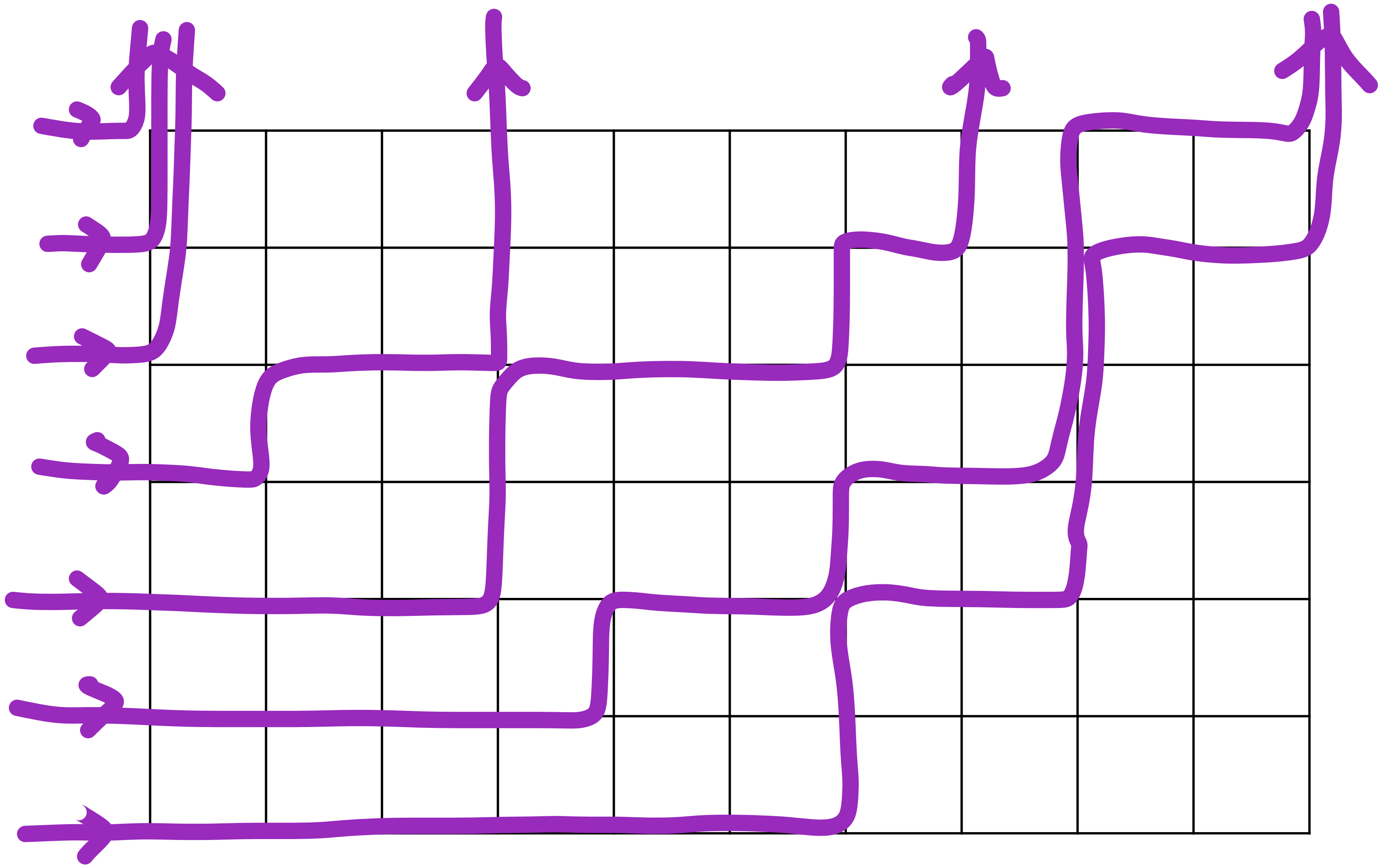
6) **Most importantly: Cool identity = interesting!!!**

Higher spin six vertex model



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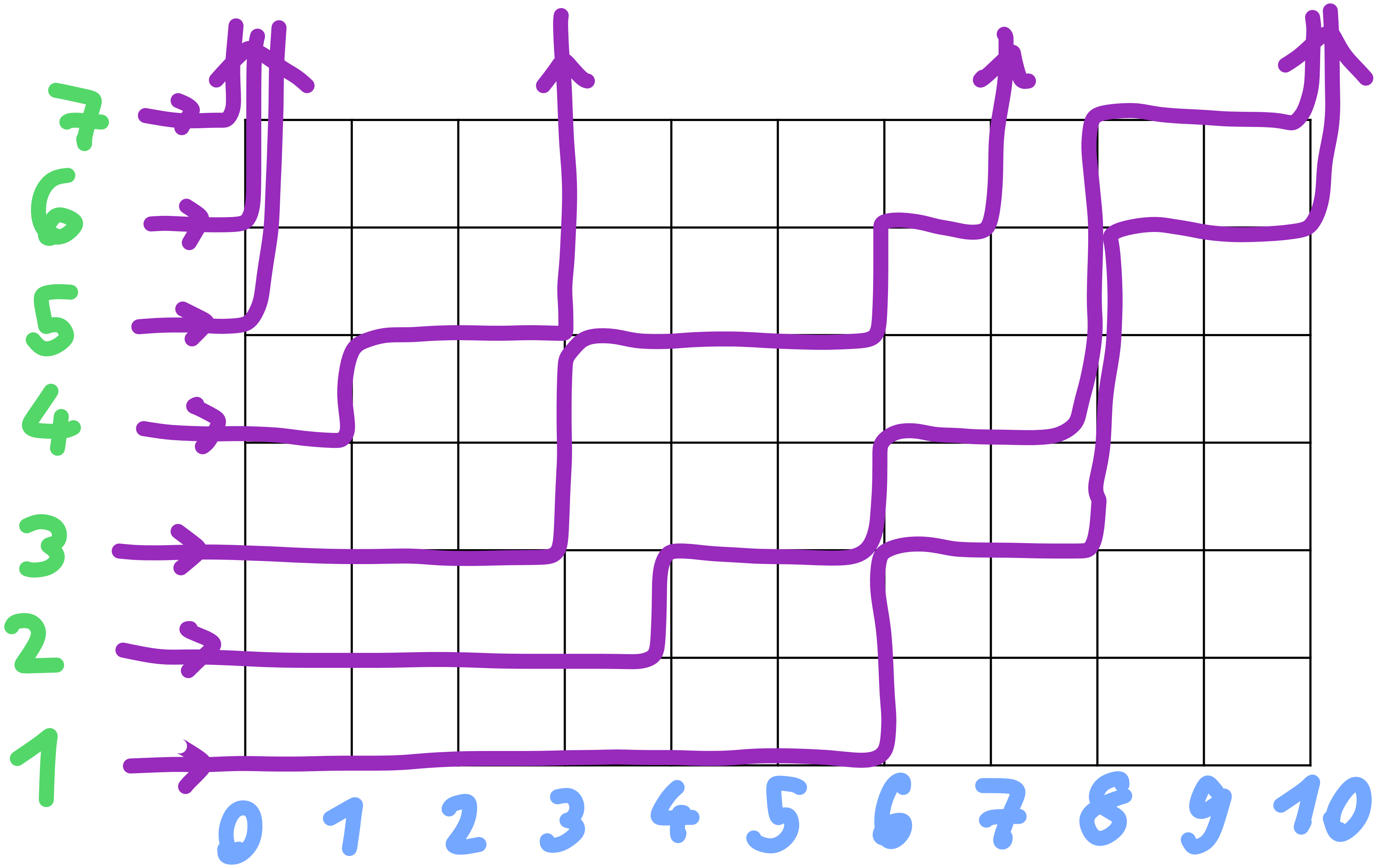

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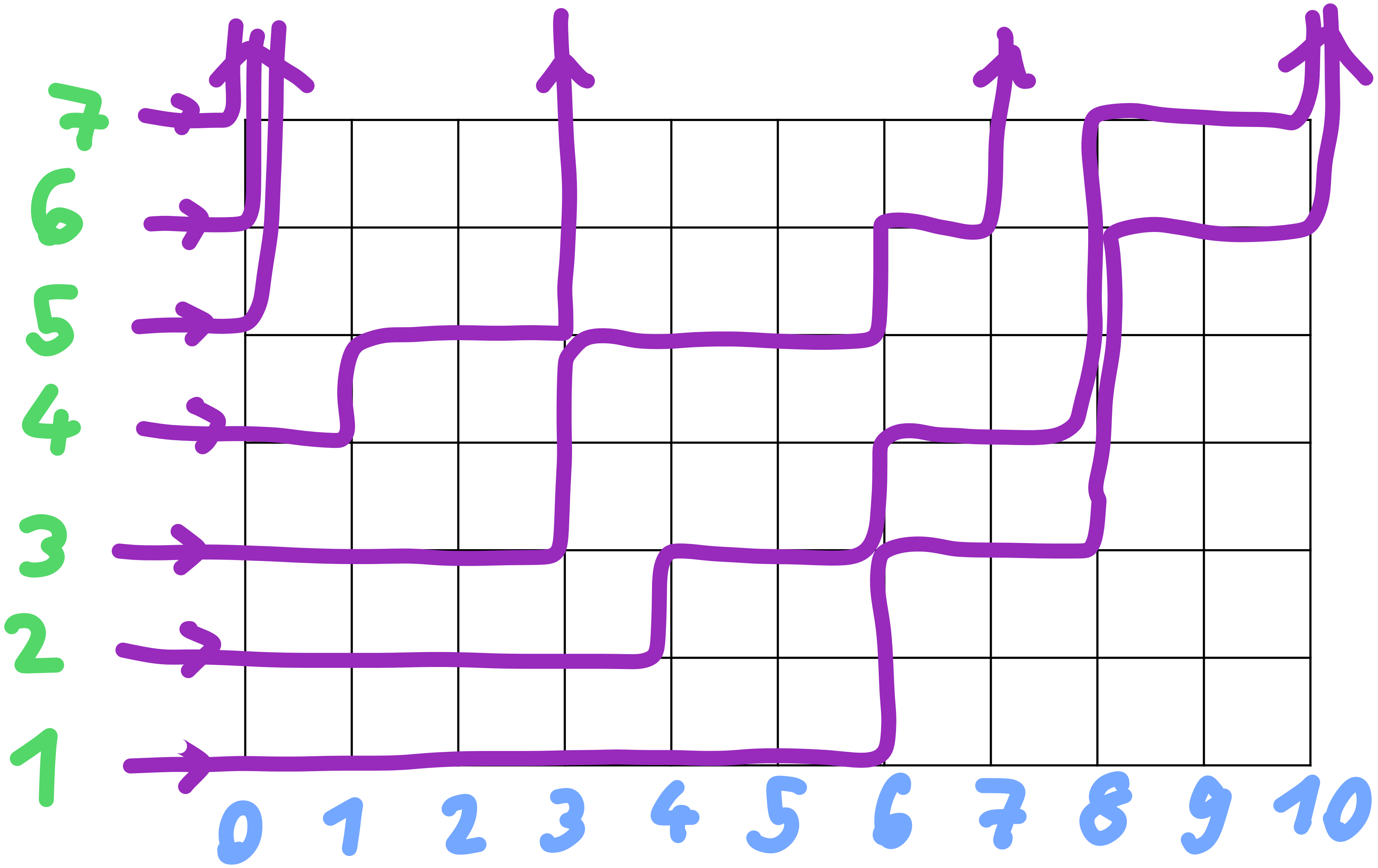

top boundary condition:

$\lambda = (10, 10, 7, 3, 0, 0, 0)$

left boundary condition:

$\longrightarrow$

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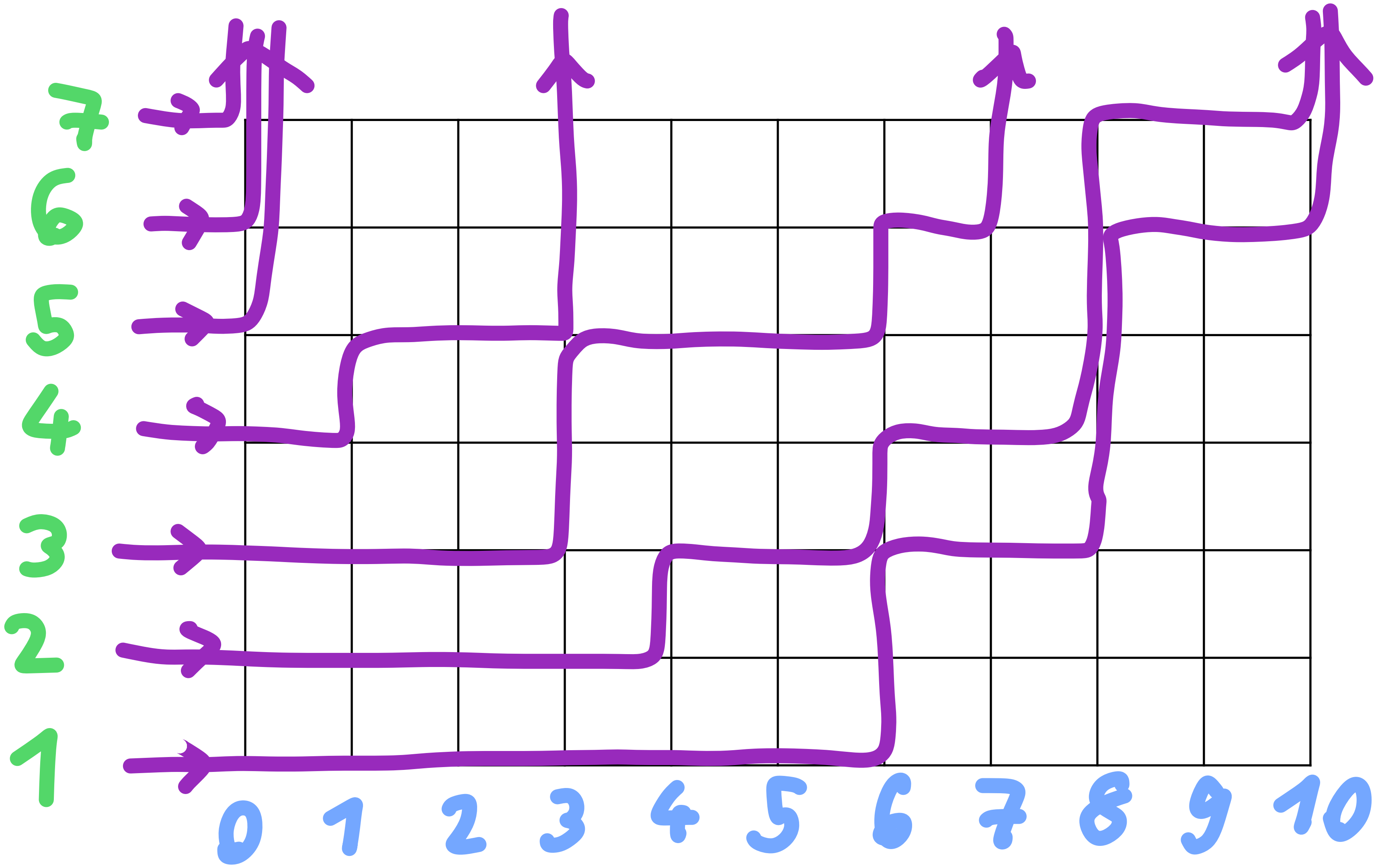

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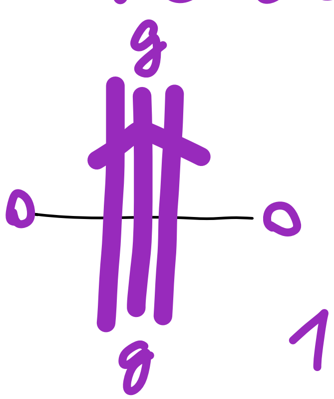
left boundary condition:

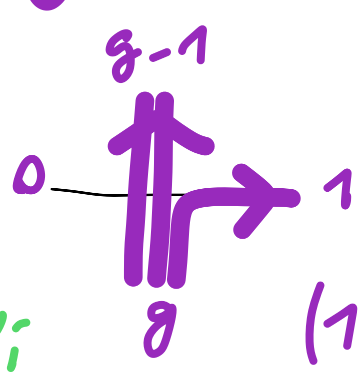
$\rightarrow$

Weight of collection of paths is product of weight of vertices

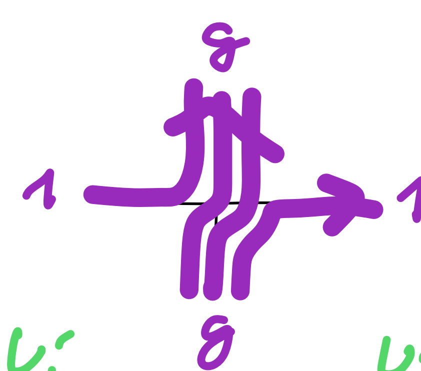
Vertex weights in position (i, j):



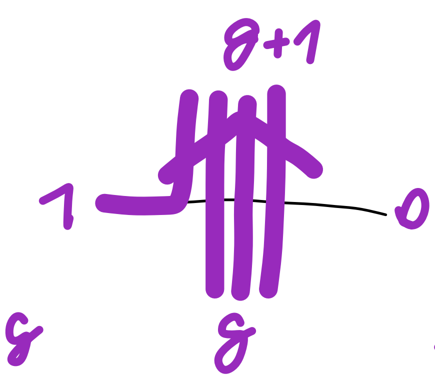
$$\frac{1 - s_j q^g v_i}{1 - s_j v_i}$$



$$\frac{(1 - s_j^2 q^{g-1}) v_i}{1 - s_j v_i}$$



$$\frac{v_i - s_j q^g}{1 - s_j v_i}$$



$$\frac{1 - q^{g+1}}{1 - s_j v_i}$$

Fully inhomogeneous spin Hall-Littlewood symmetric rational functions

Def: [Borodin, Petrov 2018]
$\lambda = (\lambda_1, \dots, \lambda_n)$ a partition with parts potentially zero.

$F_\lambda(v_1, \dots, v_n \mid s_0, s_1, \dots)$ is the generating function of all possible collections of n up-right paths in higher spin six vertex model with top boundary condition λ.

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Thm: [Borodin, Petrov 2018]

$$F_\lambda(u_1, \dots, u_n \mid s_0, s_1, \dots) = \frac{\operatorname{ASym}_{u_1, \dots, u_n}\left[\prod_{1 \le i < j \le n}(u_i - q u_j) \prod_{i=1}^{n}\left(\frac{1-q}{1 - s_{\lambda_i} u_i} \prod_{j=0}^{\lambda_i - 1} \frac{u_i - s_j}{1 - s_j u_i}\right)\right]}{\prod_{1 \le i < j \le n}(u_i - u_j)}$$

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Thm: [Borodin, Petrov 2018]

$$\mathrm{ASym}_{u_1,\dots,u_n}\left[f(u_1,\dots,u_n)\right] := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot f(u_{\sigma(1)}, \dots, u_{\sigma(n)})$$

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Idea of proof

$$H_1(u_1,\dots,u_n\,|\,s_0,s_1,\dots) \overset{\text{def}}{=} \sum_{\lambda_1\geq\dots\geq\lambda_n\geq 0} \prod_{r\geq 0} \frac{(-s_r;q)_{m_r(\lambda)}}{(q;q)_{m_r(\lambda)}} F_\lambda(u_1,\dots,u_n\,|\,s_0,s_1,\dots)$$

$$H_2^{(\gamma)}(u_1,\dots,u_n\,|\,s_0,s_1,\dots) \overset{\text{def}}{=}$$

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$$H_1(u\,|\,s_0,s_1,\dots) = \sum_{l\geq 0}\sum_{T\subsetneq[n]} (-s_l;q)_{n-|T|} \prod_{i\in T}(u_i-s_l) \prod_{i=1}^{n}\left(\frac{1}{1-s_l u_i}\prod_{j=0}^{l-1}\frac{u_i-s_j}{1-s_j u_i}\right)$$
$$\times \prod_{i\in T,\, j\in T^c} \frac{u_i-q u_j}{u_i-u_j} H_1(u_T\,|\,s_{l+1},s_{l+2},\dots)$$

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$$H_1(u_1,\dots,u_n \mid s_0,s_1,\dots) \overset{\text{def}}{=} \sum_{\lambda_1\geq\dots\geq\lambda_n\geq 0} \prod_{r\geq 0} \frac{(-s_r;q)_{m_r(\lambda)}}{(q;q)_{m_r(\lambda)}} F_\lambda(u_1,\dots,u_n \mid s_0,s_1,\dots)$$

$$H_2^{(\gamma)}(u_1,\dots,u_n \mid s_0,s_1,\dots) \overset{\text{def}}{=}$$

$$\sum_{\lambda_1\geq\dots\geq\lambda_n\geq 0} \frac{(-\gamma q^{1/2};q^{1/2})_{m_0(\lambda)}(-\gamma^{-1}s_0;q^{1/2})_{m_0(\lambda)}}{(q;q)_{m_0(\lambda)}} \prod_{r\geq 1} \frac{(-s_r;q^{1/2})_{m_r(\lambda)}}{(q^{1/2};q^{1/2})_{m_r(\lambda)}} F_\lambda(u_1,\dots,u_n \mid s_0,s_1,\dots)$$

$$H_1(\mathbf{u} \mid s_0,s_1,\dots) = \sum_{l\geq 0} \sum_{T\subsetneq[n]} (-s_l;q)_{n-|T|} \prod_{i\in T}(u_i-s_l) \prod_{i=1}^{n}\left(\frac{1}{1-s_l u_i}\prod_{j=0}^{l-1}\frac{u_i-s_j}{1-s_j u_i}\right)$$
$$\times \prod_{i\in T, j\in T^c} \frac{u_i-q u_j}{u_i-u_j} H_1(\mathbf{u}_T \mid s_{l+1},s_{l+2},\dots)$$

$$H_2^{(\gamma)}(\mathbf{u} \mid s_0,s_1,\dots) = \sum_{l\geq 0} \sum_{T\subsetneq[n]} (-\gamma^{[l=0]}q^{1/2};q^{1/2})_{n-|T|} (-\gamma^{-[l=0]}s_l;q^{1/2})_{n-|T|} \prod_{i\in T}(u_i-s_l)$$
$$\times \prod_{i=1}^{n}\left(\frac{1}{1-s_l u_i}\prod_{j=0}^{l-1}\frac{u_i-s_j}{1-s_j u_i}\right) \prod_{i\in T, j\in T^c} \frac{u_i-q u_j}{u_i-u_j} H_2^{(1)}(\mathbf{u}_T \mid s_{l+1},s_{l+2},\dots)$$

Idea of proof

$$H_1(u_1,\dots,u_n\,|\,s_0,s_1,\dots) \overset{\text{def}}{=} \sum_{\lambda_1\geq\dots\geq\lambda_n\geq 0} \prod_{r\geq 0} \frac{(-s_r;q)_{m_r(\lambda)}}{(q;q)_{m_r(\lambda)}} F_\lambda(u_1,\dots,u_n\,|\,s_0,s_1,\dots)$$

$$H_2^{(\gamma)}(u_1,\dots,u_n\,|\,s_0,s_1,\dots) \overset{\text{def}}{=}$$

$$\sum_{\lambda_1\geq\dots\geq\lambda_n\geq 0} \frac{(-\gamma q^{1/2};q^{1/2})_{m_0(\lambda)}(-\gamma^{-1}s_0;q^{1/2})_{m_0(\lambda)}}{(q;q)_{m_0(\lambda)}} \prod_{r\geq 1} \frac{(-s_r;q^{1/2})_{m_r(\lambda)}}{(q^{1/2};q^{1/2})_{m_r(\lambda)}} F_\lambda(u_1,\dots,u_n\,|\,s_0,s_1,\dots)$$

$$H_1(\mathbf{u}\,|\,s_0,s_1,\dots) = \sum_{l\geq 0}\sum_{T\subsetneq[n]} (-s_l;q)_{n-|T|} \prod_{i\in T}(u_i-s_l) \prod_{i=1}^{n}\left(\frac{1}{1-s_l u_i}\prod_{j=0}^{l-1}\frac{u_i-s_j}{1-s_j u_i}\right)$$
$$\times \prod_{i\in T,\, j\in T^c} \frac{u_i-qu_j}{u_i-u_j}\, H_1(\mathbf{u}_T\,|\,s_{l+1},s_{l+2},\dots)$$

$$H_2^{(\gamma)}(\mathbf{u}\,|\,s_0,s_1,\dots) = \sum_{l\geq 0}\sum_{T\subsetneq[n]} (-\gamma^{[l=0]}q^{\frac{1}{2}};q^{\frac{1}{2}})_{n-|T|}\,(-\gamma^{-[l=0]}s_l;q^{\frac{1}{2}})_{n-|T|} \prod_{i\in T}(u_i-s_l)$$
$$\times \prod_{i=1}^{n}\left(\frac{1}{1-s_l u_i}\prod_{j=0}^{l-1}\frac{u_i-s_j}{1-s_j u_i}\right) \prod_{i\in T,\, j\in T^c} \frac{u_i-qu_j}{u_i-u_j}\, H_2^{(1)}(\mathbf{u}_T\,|\,s_{l+1},s_{l+2},\dots)$$

Idea of proof

$$H_1(v_1,\dots,v_n \mid s_0,s_1,\dots) \overset{\text{def}}{=} \sum_{\lambda_1\geq\dots\geq\lambda_n\geq 0} \prod_{r\geq 0} \frac{(-s_r;q)_{m_r(\lambda)}}{(q;q)_{m_r(\lambda)}} F_\lambda(v_1,\dots,v_n \mid s_0,s_1,\dots)$$

$$H_2^{(\gamma)}(v_1,\dots,v_n \mid s_0,s_1,\dots) \overset{\text{def}}{=}$$

$$\sum_{\lambda_1\geq\dots\geq\lambda_n\geq 0} \frac{(-\gamma q^{1/2};q^{1/2})_{m_0(\lambda)}(-\gamma^{-1}s_0;q^{1/2})_{m_0(\lambda)}}{(q;q)_{m_0(\lambda)}} \prod_{r\geq 1} \frac{(-s_r;q^{1/2})_{m_r(\lambda)}}{(q^{1/2};q^{1/2})_{m_r(\lambda)}} F_\lambda(v_1,\dots,v_n \mid s_0,s_1,\dots)$$

$$H_1(\mathbf{v} \mid s_0,s_1,\dots) = \sum_{l\geq 0}\sum_{T\subsetneq[n]} (-s_l;q)_{n-|T|} \prod_{i\in T}(v_i-s_l) \prod_{i=1}^{n}\left(\frac{1}{1-s_l v_i}\prod_{j=0}^{l-1}\frac{v_i-s_j}{1-s_j v_i}\right)$$

$$\times \prod_{i\in T, j\in T^c} \frac{v_i-qv_j}{v_i-v_j} H_1(\mathbf{v}_T \mid s_{l+1},s_{l+2},\dots)$$

$$H_2^{(\gamma)}(\mathbf{v} \mid s_0,s_1,\dots) = \sum_{l\geq 0}\sum_{T\subsetneq[n]} (-\gamma^{[l=0]}q^{1/2};q^{1/2})_{n-|T|}(-\gamma^{-[l=0]}s_l;q^{1/2})_{n-|T|} \prod_{i\in T}(v_i-s_l)$$

$$\times \prod_{i=1}^{n}\left(\frac{1}{1-s_l v_i}\prod_{j=0}^{l-1}\frac{v_i-s_j}{1-s_j v_i}\right) \prod_{i\in T, j\in T^c} \frac{v_i-qv_j}{v_i-v_j} H_2^{(1)}(\mathbf{v}_T \mid s_{l+1},s_{l+2},\dots)$$

Idea of proof

$$H_1(v_1,\dots,v_n\,|\,s_0,s_1,\dots)\overset{\text{def}}{=}\sum_{\lambda_1\geq\dots\geq\lambda_n\geq 0}\prod_{r\geq 0}\frac{(-s_r;q)_{m_r(\lambda)}}{(q;q)_{m_r(\lambda)}}F_\lambda(v_1,\dots,v_n\,|\,s_0,s_1,\dots)$$

$$H_2^{(1)}(v_1,\dots,v_n\,|\,s_0,s_1,\dots)\overset{\text{def}}{=}\sum_{\lambda_1\geq\dots\geq\lambda_n\geq 0}\prod_{r\geq 1}\frac{(-s_r;q^{1/2})_{m_r(\lambda)}}{(q^{1/2};q^{1/2})_{m_r(\lambda)}}F_\lambda(v_1,\dots,v_n\,|\,s_0,s_1,\dots)$$

$$H_1(\mathbf{v}\,|\,s_0,s_1,\dots)=\sum_{l\geq 0}\sum_{T\subsetneq[n]}(-s_l;q)_{n-|T|}\prod_{i\in T}(v_i-s_l)\prod_{i=1}^{n}\left(\frac{1}{1-s_lv_i}\prod_{j=0}^{l-1}\frac{v_i-s_j}{1-s_jv_i}\right)$$
$$\times\prod_{i\in T,\,j\in T^c}\frac{v_i-qv_j}{v_i-v_j}\,H_1(\mathbf{v}_T\,|\,s_{l+1},s_{l+2},\dots)$$

$$H_2^{(1)}(\mathbf{v}\,|\,s_0,s_1,\dots)=\sum_{l\geq 0}\sum_{T\subsetneq[n]}(-q^{\frac12};q^{\frac12})_{n-|T|}(-s_l;q^{\frac12})_{n-|T|}\prod_{i\in T}(v_i-s_l)$$
$$\times\prod_{i=1}^{n}\left(\frac{1}{1-s_lv_i}\prod_{j=0}^{l-1}\frac{v_i-s_j}{1-s_jv_i}\right)\prod_{i\in T,\,j\in T^c}\frac{v_i-qv_j}{v_i-v_j}\,H_2^{(1)}(\mathbf{v}_T\,|\,s_{l+1},s_{l+2},\dots)$$

Idea of proof

$$H_1(v_1,\dots,v_n\,|\,s_0,s_1,\dots) \overset{\text{def}}{=} \sum_{\lambda_1\geq\dots\geq\lambda_n\geq 0} \prod_{r\geq 0} \frac{(-s_r;q)_{m_r(\lambda)}}{(q;q)_{m_r(\lambda)}} F_\lambda(v_1,\dots,v_n\,|\,s_0,s_1,\dots)$$

$$H_2^{(1)}(v_1,\dots,v_n\,|\,s_0,s_1,\dots) \overset{\text{def}}{=} \sum_{\lambda_1\geq\dots\geq\lambda_n\geq 0} \prod_{r\geq 1} \frac{(-s_r;q^{1/2})_{m_r(\lambda)}}{(q^{1/2};q^{1/2})_{m_r(\lambda)}} F_\lambda(v_1,\dots,v_n\,|\,s_0,s_1,\dots)$$

Plug in RHSs for H_1, $H_2^{(1)}$, which are independent of $s_0,s_1,\dots$!!!

$$H_1(\mathbf{v}\,|\,s_0,s_1,\dots) = \sum_{l\geq 0}\sum_{T\subsetneq[n]} (-s_l;q)_{n-|T|} \prod_{i\in T}(v_i-s_l) \prod_{i=1}^{n}\left(\frac{1}{1-s_l v_i}\prod_{j=0}^{l-1}\frac{v_i-s_j}{1-s_j v_i}\right)$$

The only dependence on "s" & l is:

$$\times \prod_{i\in T,\, j\in T^c} \frac{v_i-qv_j}{v_i-v_j} H_1(\mathbf{v}_T\,|\,s_{l+1},s_{l+2},\dots)$$

$$H_2^{(1)}(\mathbf{v}\,|\,s_0,s_1,\dots) = \sum_{l\geq 0}\sum_{T\subsetneq[n]} (-q^{1/2};q^{1/2})_{n-|T|}(-s_l;q^{1/2})_{n-|T|} \prod_{i\in T}(v_i-s_l)$$

$$\times \prod_{i=1}^{n}\left(\frac{1}{1-s_l v_i}\prod_{j=0}^{l-1}\frac{v_i-s_j}{1-s_j v_i}\right) \prod_{i\in T,\, j\in T^c}\frac{v_i-qv_j}{v_i-v_j} H_2^{(1)}(\mathbf{v}_T\,|\,s_{l+1},s_{l+2},\dots)$$

Idea of proof

Now infinite sum can be evaluated! We assumed $s_j = s_p \ \forall j \geq p$, so:

$$\sum_{l\geq 0} (-s_l;x)_{n-|T|} \prod_{i\in T}(v_i - s_l) \prod_{i=1}^{n}\left(\frac{1}{1-s_l v_i}\prod_{j=0}^{l-1}\frac{v_i - s_j}{1-s_j v_i}\right)$$

$$= \sum_{l=0}^{p-1} (-s_l;x)_{n-|T|} \prod_{i\in T}(v_i - s_l) \prod_{i=1}^{n}\left(\frac{1}{1-s_l v_i}\prod_{j=0}^{l-1}\frac{v_i - s_j}{1-s_j v_i}\right)$$

$$+ \boxed{\frac{(-s_p;x)_{n-|T|} \prod_{i\in T}(v_i - s_p) \prod_{i=1}^{n}\left(\frac{1}{1-s_p v_i}\prod_{j=0}^{p-1}\frac{v_i - s_p}{1-s_p v_i}\right)}{1 - \prod_{i=1}^{n}\frac{v_i - s_p}{1-s_p v_i}}}$$

Idea of proof

Now infinite sum can be evaluated! We assumed $s_j = s_p \ \forall j \geq p$, so:

$$\sum_{l \geq 0} (-s_l;x)_{n-|T|} \prod_{i \in T} (v_i - s_l) \prod_{i=1}^{n} \left(\frac{1}{1 - s_l v_i} \boxed{\prod_{j=0}^{l-1} \frac{v_i - s_j}{1 - s_j v_i}} \right)$$

$$= \sum_{l=0}^{p-1} (-s_l;x)_{n-|T|} \prod_{i \in T} (v_i - s_l) \prod_{i=1}^{n} \left(\frac{1}{1 - s_l v_i} \prod_{j=0}^{l-1} \frac{v_i - s_j}{1 - s_j v_i} \right)$$

$$+ \frac{(-s_p;x)_{n-|T|} \prod_{i \in T} (v_i - s_p) \prod_{i=1}^{n} \left(\frac{1}{1 - s_p v_i} \prod_{j=0}^{p-1} \frac{v_i - s_p}{1 - s_p v_i} \right)}{1 - \prod_{i=1}^{n} \frac{v_i - s_p}{1 - s_p v_i}}$$

Idea of proof

Now infinite sum can be evaluated! We assumed $s_j = s_p \ \forall j \geq p$, so:

$$\sum_{l \geq 0} (-s_l; x)_{n-|T|} \prod_{i \in T} (v_i - s_l) \prod_{i=1}^{n} \left(\frac{1}{1 - s_l v_i} \prod_{j=0}^{l-1} \frac{v_i - s_j}{1 - s_j v_i} \right)$$

$$= \sum_{l=0}^{p-1} (-s_l; x)_{n-|T|} \prod_{i \in T} (v_i - s_l) \prod_{i=1}^{n} \left(\frac{1}{1 - s_l v_i} \prod_{j=0}^{l-1} \frac{v_i - s_j}{1 - s_j v_i} \right)$$

$$+ \frac{(-s_p; x)_{n-|T|} \prod_{i \in T} (v_i - s_p) \prod_{i=1}^{n} \left(\frac{1}{1 - s_p v_i} \prod_{j=0}^{p-1} \frac{v_i - s_p}{1 - s_p v_i} \right)}{1 - \prod_{i=1}^{n} \frac{v_i - s_p}{1 - s_p v_i}}$$

Multiply by this!

Idea of proof

Now infinite sum can be evaluated! We assumed $s_j = s_p \; \forall j \geq p$, so:

$$\left(1 - \prod_{i=1}^{n} \frac{v_i - s_p}{1 - s_p v_i}\right) \sum_{l \geq 0} (-s_l; x)_{n-|T|} \prod_{i \in T} (v_i - s_l) \prod_{i=1}^{n} \left(\frac{1}{1 - s_l v_i} \prod_{j=0}^{l-1} \frac{v_i - s_j}{1 - s_j v_i} \right)$$

$$= \sum_{l=0}^{p} (-s_l; x)_{n-|T|} \prod_{i \in T} (v_i - s_l) \prod_{i=1}^{n} \left(\frac{1}{1 - s_l v_i} \prod_{j=0}^{l-1} \frac{v_i - s_j}{1 - s_j v_i} \right)$$

$$- \prod_{i=1}^{n} \frac{v_i - s_p}{1 - s_p v_i} \sum_{l=0}^{p-1} (-s_l; x)_{n-|T|} \prod_{i \in T} (v_i - s_l) \prod_{i=1}^{n} \left(\frac{1}{1 - s_l v_i} \prod_{j=0}^{l-1} \frac{v_i - s_j}{1 - s_j v_i} \right)$$

Idea of proof

Now infinite sum can be evaluated! We assumed $s_j = s_p \ \forall j \geq p$, so:

$$\left(1 - \prod_{i=1}^{n} \frac{v_i - s_p}{1 - s_p v_i}\right) \sum_{l \geq 0} (-s_l; x)_{n-|T|} \prod_{i \in T} (v_i - s_l) \prod_{i=1}^{n} \left(\frac{1}{1 - s_l v_i} \prod_{j=0}^{l-1} \frac{v_i - s_j}{1 - s_j v_i} \right)$$

$$= \sum_{l=0}^{p} (-s_l; x)_{n-|T|} \prod_{i \in T} (v_i - s_l) \prod_{i=1}^{n} \left(\frac{1}{1 - s_l v_i} \prod_{j=0}^{l-1} \frac{v_i - s_j}{1 - s_j v_i} \right)$$

$$- \prod_{i=1}^{n} \frac{v_i - s_p}{1 - s_p v_i} \sum_{l=0}^{p-1} (-s_l; x)_{n-|T|} \prod_{i \in T} (v_i - s_l) \prod_{i=1}^{n} \left(\frac{1}{1 - s_l v_i} \prod_{j=0}^{l-1} \frac{v_i - s_j}{1 - s_j v_i} \right)$$

This is equivalent to:

$$\prod_{i=1}^{n} \prod_{j=0}^{k-1} \frac{v_i - s_j}{1 - s_j v_i} \left(1 - \prod_{i=1}^{n} \frac{v_i - s_p}{1 - s_p v_i}\right) \sum_{l \geq 0} (-s_l; x)_{n-|T|} \prod_{i \in T} (v_i - s_l) \prod_{i=1}^{n} \left(\frac{1}{1 - s_l v_i} \prod_{j=0}^{l-1} \frac{v_i - s_j}{1 - s_j v_i} \right)$$

$$= (-s_k; x)_{n-|T|} \prod_{i \in T} (v_i - s_k) \prod_{i=1}^{n} \left(\frac{1}{1 - s_k v_i} \prod_{j=0}^{k-1} \frac{v_i - s_j}{1 - s_j v_i} \right)$$

for arbitrary $k \geq 0$. No sum anymore!

Idea of proof

After a few further manipulations we obtain two polynomial identities:

$$\prod_{i=1}^{n}(1-sv_i)\prod_{1\le i<j\le n}(1-qv_iv_j)(v_i-v_j)=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T, j\in T^c \mid i>j\}}(-s;q)_{n-|T|}$$

$$\times\prod_{i\in T^c}(1-v_i)\prod_{i\in T}(v_i-s)\prod_{\substack{i,j\in T^c\\ i<j}}(1-v_iv_j)(v_i-v_j)\prod_{i\in T, j\in T^c}(v_i-qv_j)(1-v_iv_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qv_iv_j)(v_i-v_j)$$

and

$$\prod_{i=1}^{n}(1+q^{1/2})(1-sv_i)\prod_{1\le i<j\le n}(1-qv_iv_j)(1-v_iv_j)\,\mathrm{Pf}\left(M^{(1)}\right)=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T, j\in T^c \mid i>j\}}$$

$$\times(-s;q^{1/2})_{n-|T|}(-q^{1/2};q^{1/2})_{n-|T|}\prod_{i\in T^c}(1-v_i)\prod_{i\in T}(v_i-s)(1+q^{1/2})\prod_{\substack{i,j\in T^c\\ i<j}}(1-v_iv_j)(v_i-v_j)$$

$$\times\prod_{i\in T, j\in T^c}(v_i-qv_j)(1-v_iv_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qv_iv_j)(v_i-v_j)\,\mathrm{Pf}\left(M_T^{(1)}\right)$$

Idea of proof

After a few further manipulations we obtain two polynomial identities:

$$\prod_{i=1}^{n}(1-su_i)\prod_{1\le i<j\le n}(1-qu_iu_j)(u_i-u_j)=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T,j\in T^c\mid i>j\}}(-s;q)_{n-|T|}$$

$$\times\prod_{i\in T^c}(1-u_i)\prod_{i\in T}(u_i-s)\prod_{\substack{i,j\in T^c\\ i<j}}(1-u_iu_j)(u_i-u_j)\prod_{i\in T,j\in T^c}(u_i-qu_j)(1-u_iu_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qu_iu_j)(u_i-u_j)$$

and

$$\prod_{i=1}^{n}(1+q^{\frac{1}{2}})(1-su_i)\prod_{1\le i<j\le n}(1-qu_iu_j)(1-u_iu_j)\,\mathrm{Pf}(M^{(1)})=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T,j\in T^c\mid i>j\}}$$

$$\times(-s;q^{\frac{1}{2}})_{n-|T|}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{n-|T|}\prod_{i\in T^c}(1-u_i)\prod_{i\in T}(u_i-s)(1+q^{\frac{1}{2}})\prod_{\substack{i,j\in T^c\\ i<j}}(1-u_iu_j)(u_i-u_j)$$

$$\times\prod_{i\in T,j\in T^c}(u_i-qu_j)(1-u_iu_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qu_iu_j)(u_i-u_j)\;\mathrm{Pf}(M^{(1)}_T)$$

Idea of proof

After a few further manipulations we obtain two polynomial identities:

$$\prod_{i=1}^{n}(1-sv_i)\prod_{1\le i<j\le n}(1-qv_iv_j)(v_i-v_j)=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T, j\in T^c \mid i>j\}}(-s;q)_{n-|T|}$$
$$\times\prod_{i\in T^c}(1-v_i)\prod_{i\in T}(v_i-s)\prod_{\substack{i,j\in T^c\\ i<j}}(1-v_iv_j)(v_i-v_j)\prod_{i\in T, j\in T^c}(v_i-qv_j)(1-v_iv_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qv_iv_j)(v_i-v_j)$$

and

$$\prod_{i=1}^{n}(1+q^{1/2})(1-sv_i)\prod_{1\le i<j\le n}(1-qv_iv_j)(1-v_iv_j)\,\mathrm{Pf}(M^{(1)})=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T, j\in T^c \mid i>j\}}$$
$$\times(-s;q^{1/2})_{n-|T|}\,(-q^{1/2};q^{1/2})_{n-|T|}\prod_{i\in T^c}(1-v_i)\prod_{i\in T}(v_i-s)(1+q^{1/2})\prod_{\substack{i,j\in T^c\\ i<j}}(1-v_iv_j)(v_i-v_j)$$
$$\times\prod_{i\in T, j\in T^c}(v_i-qv_j)(1-v_iv_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qv_iv_j)(v_i-v_j)\;\mathrm{Pf}(M^{(1)}_T)$$

Idea of proof

After a few further manipulations we obtain two polynomial identities:

$$\prod_{i=1}^{n}(1-sv_i)\prod_{1\le i<j\le n}(1-qv_iv_j)(v_i-v_j)=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T, j\in T^c \mid i>j\}}(-s;q)_{n-|T|}$$
$$\times\prod_{i\in T^c}(1-v_i)\prod_{i\in T}(v_i-s)\prod_{\substack{i,j\in T^c\\ i<j}}(1-v_iv_j)(v_i-v_j)\prod_{i\in T, j\in T^c}(v_i-qv_j)(1-v_iv_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qv_iv_j)(v_i-v_j)$$

and

$$\prod_{i=1}^{n}(1+q^{\frac{1}{2}})(1-sv_i)\prod_{1\le i<j\le n}(1-qv_iv_j)(1-v_iv_j)\,\mathrm{Pf}\left(M^{(1)}\right)=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T, j\in T^c \mid i>j\}}$$
$$\times(-s;q^{\frac{1}{2}})_{n-|T|}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{n-|T|}\prod_{i\in T^c}(1-v_i)\prod_{i\in T}(v_i-s)(1+q^{\frac{1}{2}})\prod_{\substack{i,j\in T^c\\ i<j}}(1-v_iv_j)(v_i-v_j)$$
$$\times\prod_{i\in T, j\in T^c}(v_i-qv_j)(1-v_iv_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qv_iv_j)(v_i-v_j)\,\mathrm{Pf}\left(M^{(1)}_{T}\right)$$

Idea of proof

After a few further manipulations we obtain two polynomial identities:

$$\prod_{i=1}^{n}(1-sv_i)\prod_{1\le i<j\le n}(1-qv_iv_j)(v_i-v_j)=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T, j\in T^c \mid i>j\}}(-s;q)_{n-|T|}$$

$$\times\prod_{i\in T^c}(1-v_i)\prod_{i\in T}(v_i-s)\prod_{\substack{i,j\in T^c\\ i<j}}(1-v_iv_j)(v_i-v_j)\prod_{i\in T, j\in T^c}(v_i-qv_j)(1-v_iv_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qv_iv_j)(v_i-v_j)$$

and

$$\prod_{i=1}^{n}(1+q^{1/2})(1-sv_i)\prod_{1\le i<j\le n}(1-qv_iv_j)(1-v_iv_j)\,\mathrm{Pf}(M^{(\gamma)})=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T, j\in T^c \mid i>j\}}$$

$$\times(-\gamma^{-1}s;q^{1/2})_{n-|T|}(-\gamma q^{1/2};q^{1/2})_{n-|T|}\prod_{i\in T^c}(1-v_i)\prod_{i\in T}(v_i-s)(1+q^{1/2})\prod_{\substack{i,j\in T^c\\ i<j}}(1-v_iv_j)(v_i-v_j)$$

$$\times\prod_{i\in T, j\in T^c}(v_i-qv_j)(1-v_iv_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qv_iv_j)(v_i-v_j)\;\mathrm{Pf}(M_T^{(1)})$$

Idea of proof

After a few further manipulations we obtain two polynomial identities:

$$\prod_{i=1}^{n}(1-sv_i)\prod_{1\le i<j\le n}(1-qv_iv_j)(v_i-v_j)=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T, j\in T^c \mid i>j\}}(-s;q)_{n-|T|}$$
$$\times\prod_{i\in T^c}(1-v_i)\prod_{i\in T}(v_i-s)\prod_{\substack{i,j\in T^c\\ i<j}}(1-v_iv_j)(v_i-v_j)\prod_{i\in T, j\in T^c}(v_i-qv_j)(1-v_iv_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qv_iv_j)(v_i-v_j)$$

and

$$\prod_{i=1}^{n}(1+q^{1/2})(1-sv_i)\prod_{1\le i<j\le n}(1-qv_iv_j)(1-v_iv_j)\,\mathrm{Pf}(M^{(\gamma)})=\sum_{T\subseteq[n]}(-1)^{\#\{i\in T, j\in T^c \mid i>j\}}$$
$$\times(-\gamma^{-1}s;q^{1/2})_{n-|T|}(-\gamma q^{1/2};q^{1/2})_{n-|T|}\prod_{i\in T^c}(1-v_i)\prod_{i\in T}(v_i-s)(1+q^{1/2})\prod_{\substack{i,j\in T^c\\ i<j}}(1-v_iv_j)(v_i-v_j)$$
$$\times\prod_{i\in T, j\in T^c}(v_i-qv_j)(1-v_iv_j)\prod_{\substack{i,j\in T\\ i<j}}(1-qv_iv_j)(v_i-v_j)\,\mathrm{Pf}(M_T^{(1)})$$

Both proven by evaluations at: $v_1=v_k, v_k^{-1}$ for $k=2,\dots,n$ & $v_1=1, s$ $\square$

The modified Robbins polynomials

$$R^{*}_{(k_1,\dots,k_n)}(x_1,\dots,x_n;u,v,w) = \frac{\mathrm{ASym}_{x_1,\dots,x_n}\left[\prod_{1\le i<j\le n}(u x_i x_j + v + w x_i)\prod_{i=1}^{n} x_i^{k_i}\right]}{\prod_{1\le i<j\le n}(x_j - x_i)}$$

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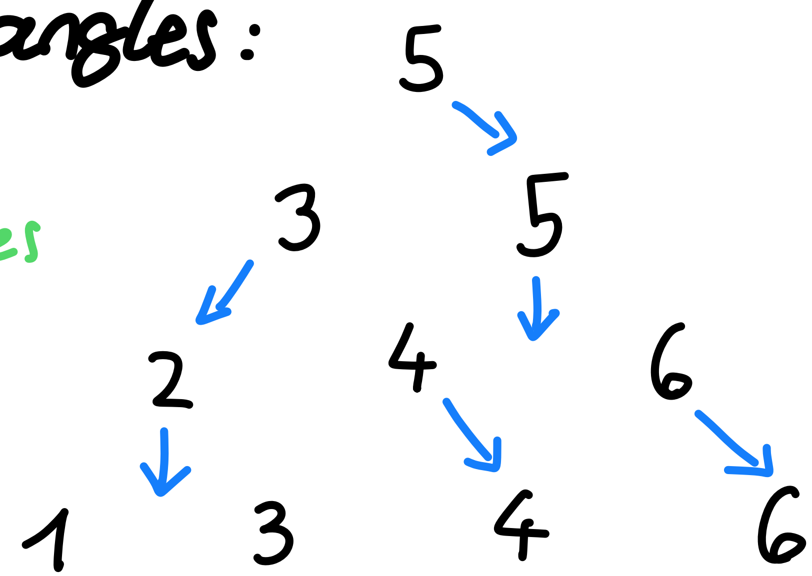
Combinatorial model: (for $k_1<k_2<\ldots$)

Down arrowed monotone triangles:



$(k_1,\ldots,k_n)$ gives bottom row!

Motto "Arrows if equality in that direction!"

The modified Robbins polynomials

$$R^{*}_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;u,v,w) = \frac{\mathrm{ASym}_{x_1,\ldots,x_n}\left[\prod_{1\le i<j\le n}(u x_i x_j + v + w x_i)\prod_{i=1}^{n} x_i^{k_i}\right]}{\prod_{1\le i<j\le n}(x_j - x_i)}$$

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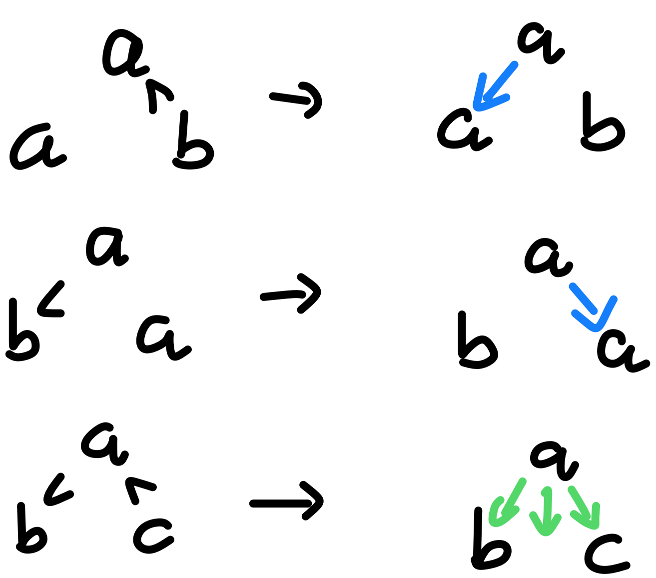

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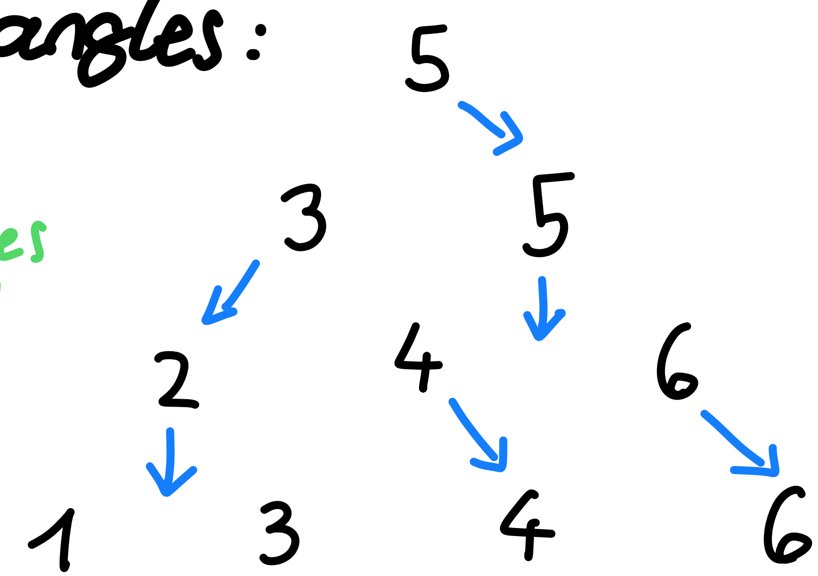

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The modified Robbins polynomials

$$R^*_{(k_1,\dots,k_n)}(x_1,\dots,x_n;u,v,w) = \frac{\mathrm{ASym}_{x_1,\dots,x_n}\left[\prod_{1\le i<j\le n}(u x_i x_j + v + w x_i)\prod_{i=1}^{n} x_i^{k_i}\right]}{\prod_{1\le i<j\le n}(x_j - x_i)}$$

Combinatorial model: (for $k_1<k_2<\dots$)

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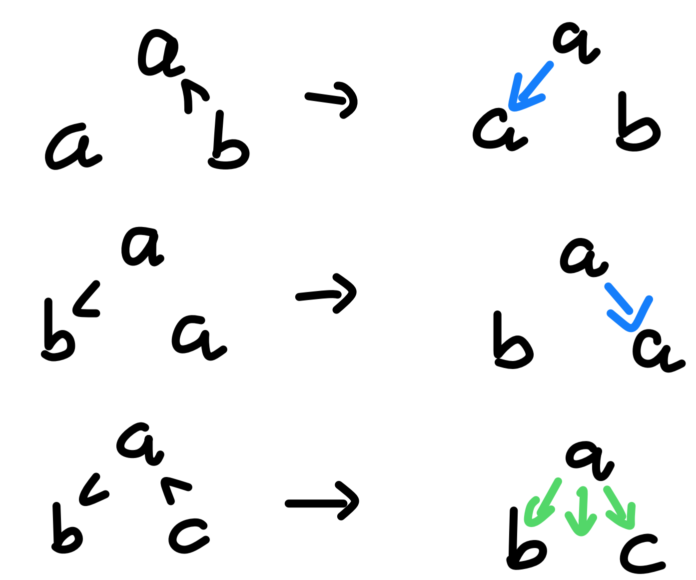
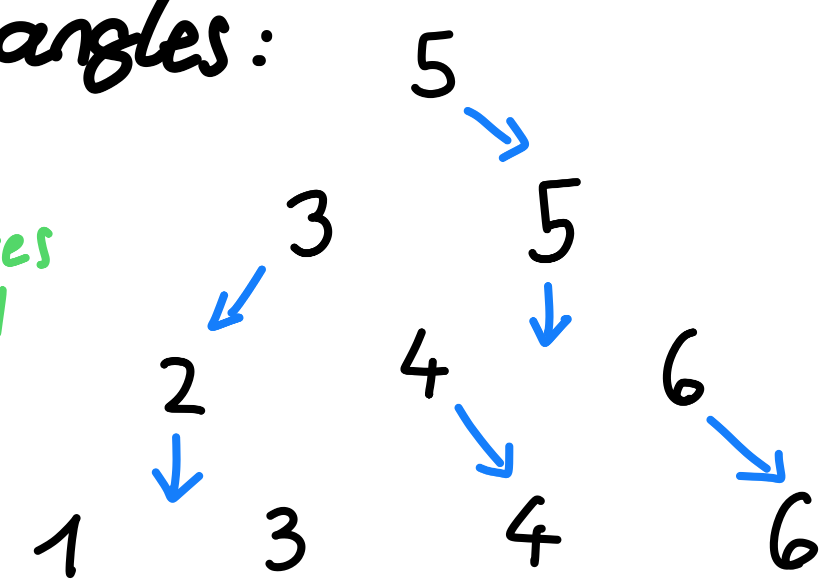




$(k_1,\dots,k_n)$ gives bottom row!

Motto "Arrows if equality in that direction!"

Weight:

$$u^{\#\searrow}\, v^{\#\swarrow}\, w^{\#\downarrow} \prod_{i=1}^{n} x_i^{\sum(\text{row } i) - \sum(\text{row } i-1) + \#\searrow \text{ in row } i-1 \;-\; \#\swarrow \text{ in row } i-1}$$

F.i. spin Hall-Littlewood s.r.f. $\longleftrightarrow$ Robbins polynomials

Among others, we have the relation for $\lambda = \lambda_1 > \lambda_2 > \ldots > \lambda_n \geq 0$:

$$F_\lambda\left(u_1,\ldots,u_n \,\middle|\, -q^{-\frac{1}{2}}\right)\Bigg|_{u_i = \frac{-q^{-\frac{1}{2}}+x_i}{1-q^{-\frac{1}{2}}x_i}}$$

$$= \frac{1}{(1-q^{-1})^{\binom{n}{2}}} \prod_{i=1}^{n} \frac{1-q+(q^{1/2}-q^{-1/2})x_i}{1-q^{-1}}$$

$$\times R^*_{\lambda^{\mathrm{rev}}}(x_1,\ldots,x_n; v,v,w),$$

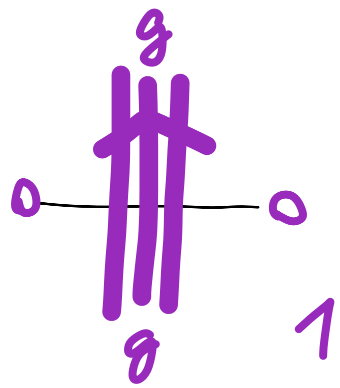
where $v = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, $w = q^{-1} - q$.

Bijection: higher spin 6VM $\leftrightarrow$ DAMTs

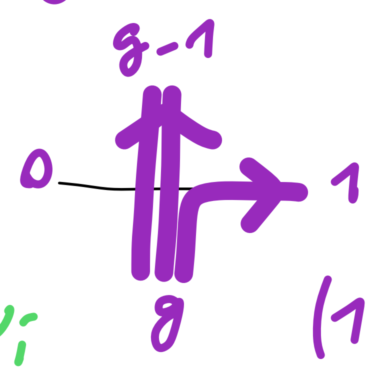
$\boxed{\text{Assume } \lambda_1 > \lambda_2 > \dots > \lambda_n \geq 0.}$

Recall the vertex weights for the higher spin 6VM:

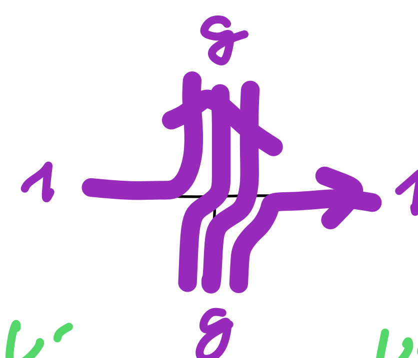
Vertex weights in position (i, j):



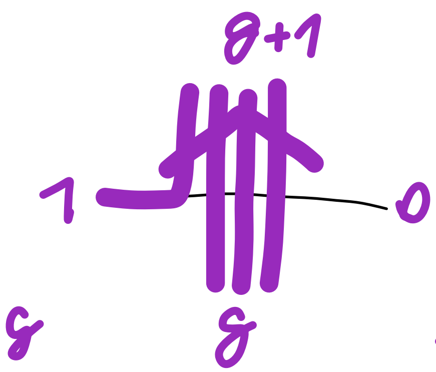
$$\frac{1 - s_j q^g v_i}{1 - s_j v_i}$$



$$\frac{(1 - s_j^2 q^{g-1}) v_i}{1 - s_j v_i}$$

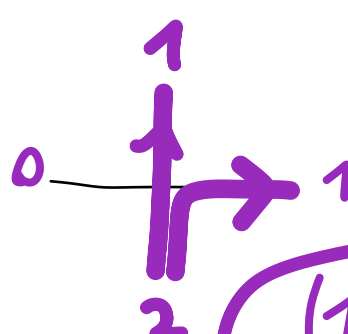


$$\frac{v_i - s_j q^g}{1 - s_j v_i}$$



$$\frac{1 - q^{g+1}}{1 - s_j v_i}$$

Setting $s = -q^{-\frac{1}{2}}$:



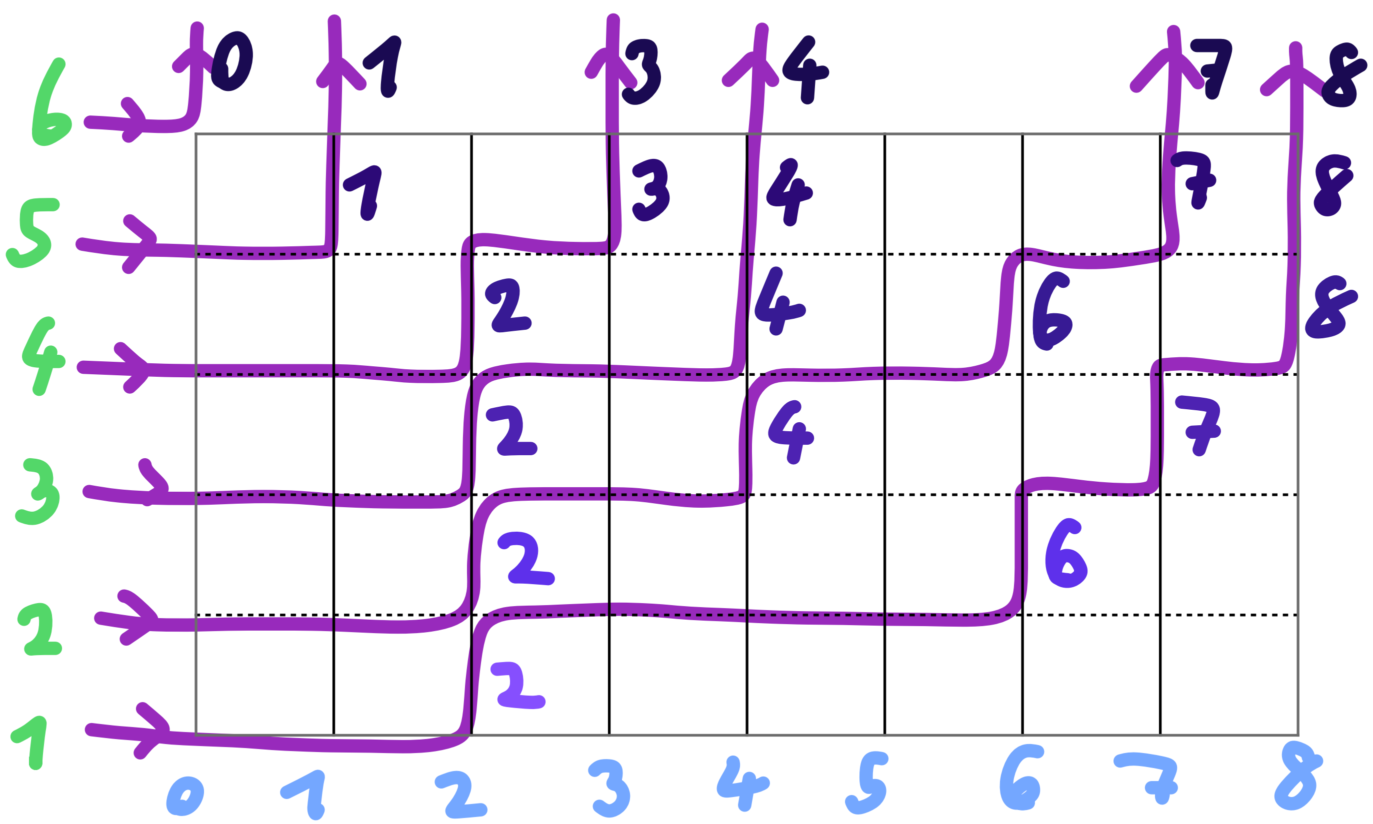
$$\frac{(1 - (-q^{-\frac{1}{2}})^2 q) v_i}{1 + q^{-\frac{1}{2}} v_i} = 0$$

$\Rightarrow$ no shared edges between any paths as top boundary exits are all different!

Bijection: higher spin 6VM $\longleftrightarrow$ DAMTs

$\boxed{\text{Assume } \lambda_1 > \lambda_2 > \ldots > \lambda_n \geq 0.}$ Set $s = q^{-\frac{1}{2}}$.

$\Rightarrow$ no shared edges between any paths



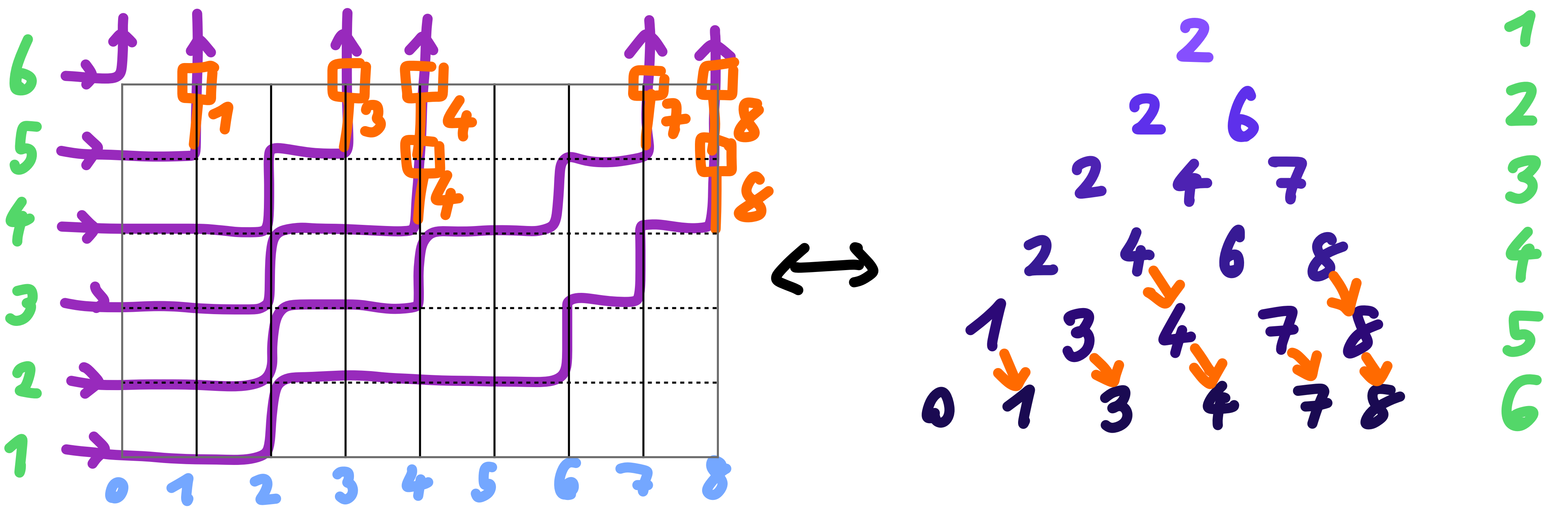
$\longleftrightarrow$

1	2					
2	2	6				
3	2	4	7			
4	2	4	6	8		
5	1	3	4	7	8	
6	0	1	3	4	7	8

a-th rightmost path in row i $\longleftrightarrow$ a-th entry in row i

Bijection: higher spin 6VM $\leftrightarrow$ DAMTs

$\boxed{\text{Assume } \lambda_1 > \lambda_2 > \ldots > \lambda_n \geq 0.}$ Set $s = q^{-\frac{1}{2}}$.

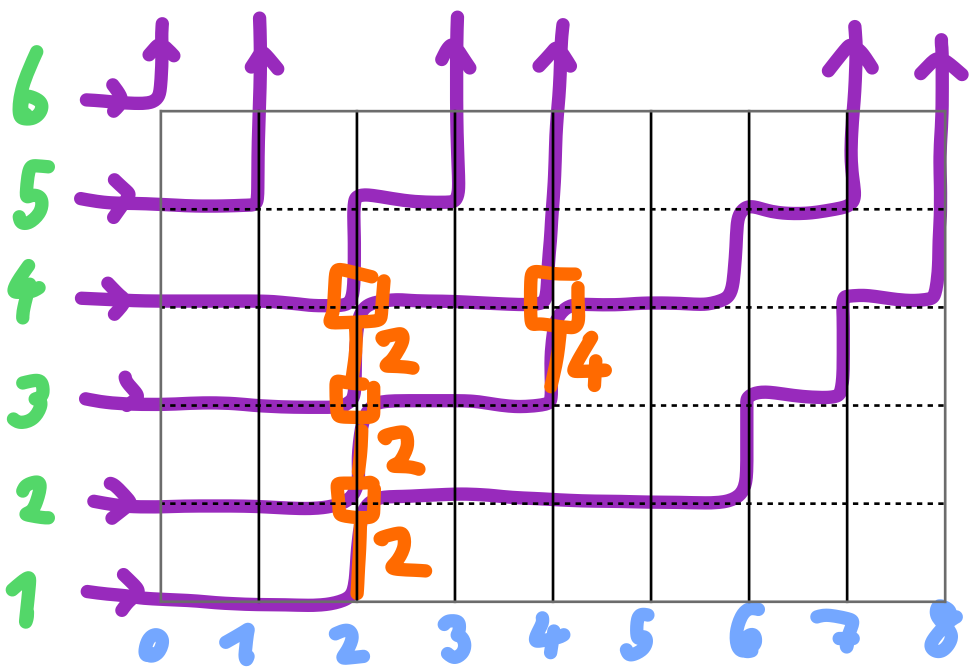


Claim: vertices $\longleftrightarrow$ $\begin{matrix} & a & \\ b & \searrow & a \end{matrix}$ entries

Proof: Assume is a-th path in row i, then it is a+1st path in row i+1. $\square$

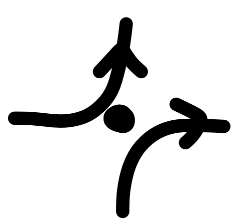
Bijection: higher spin 6VM $\leftrightarrow$ DAMTs

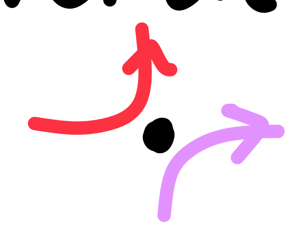
$\boxed{\text{Assume } \lambda_1 > \lambda_2 > \ldots > \lambda_n \geq 0.}$ Set $s = q^{-\frac{1}{2}}$.



$\longleftrightarrow$

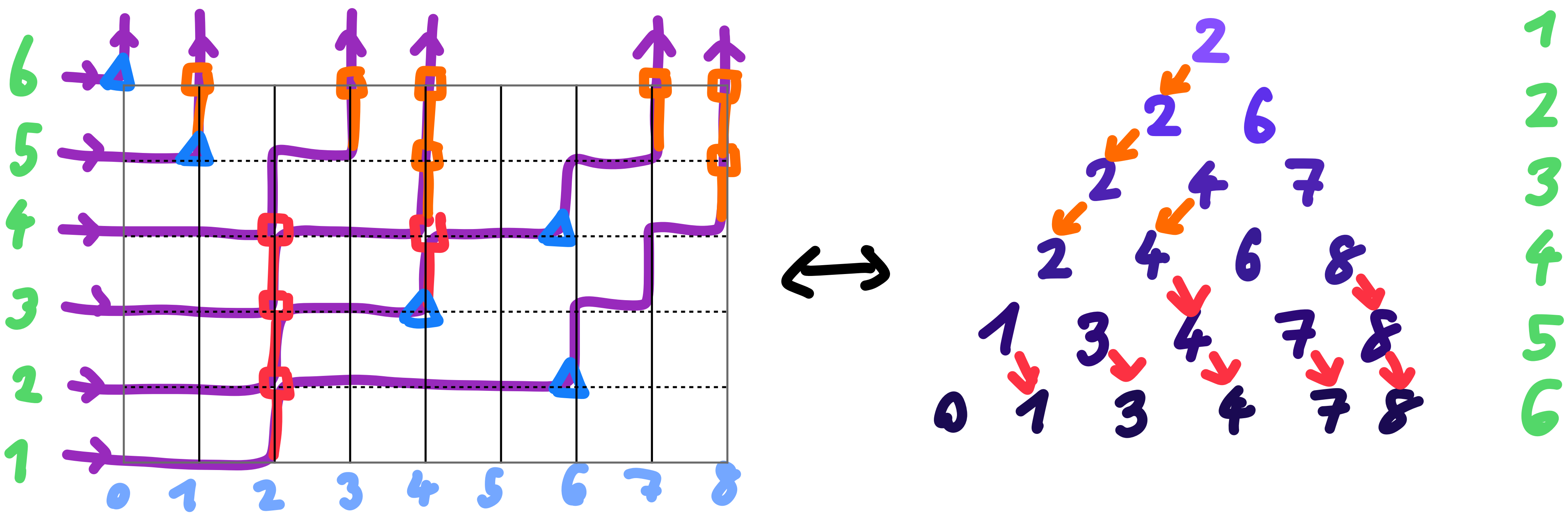
$$\begin{array}{ccccccccccccl}
 & & & & & 2 & & & & & & & 1\\
 & & & & 2 & & 6 & & & & & & 2\\
 & & & 2 & & 4 & & 7 & & & & & 3\\
 & & 2 & & 4 & & 6 & & 8 & & & & 4\\
 & 1 & & 3 & & 4 & & 7 & & 8 & & & 5\\
0 & & 1 & & 3 & & 4 & & 7 & & 8 & & 6
\end{array}$$

Claim:  vertices $\leftrightarrow$ $a \swarrow^{a}\, b$ configurations

Proof: If  in row i and pink path is a-th path in row $i-1$ then red path is a-th path in row i. $\square$

Bijection: higher spin 6VM $\longleftrightarrow$ DAMTs

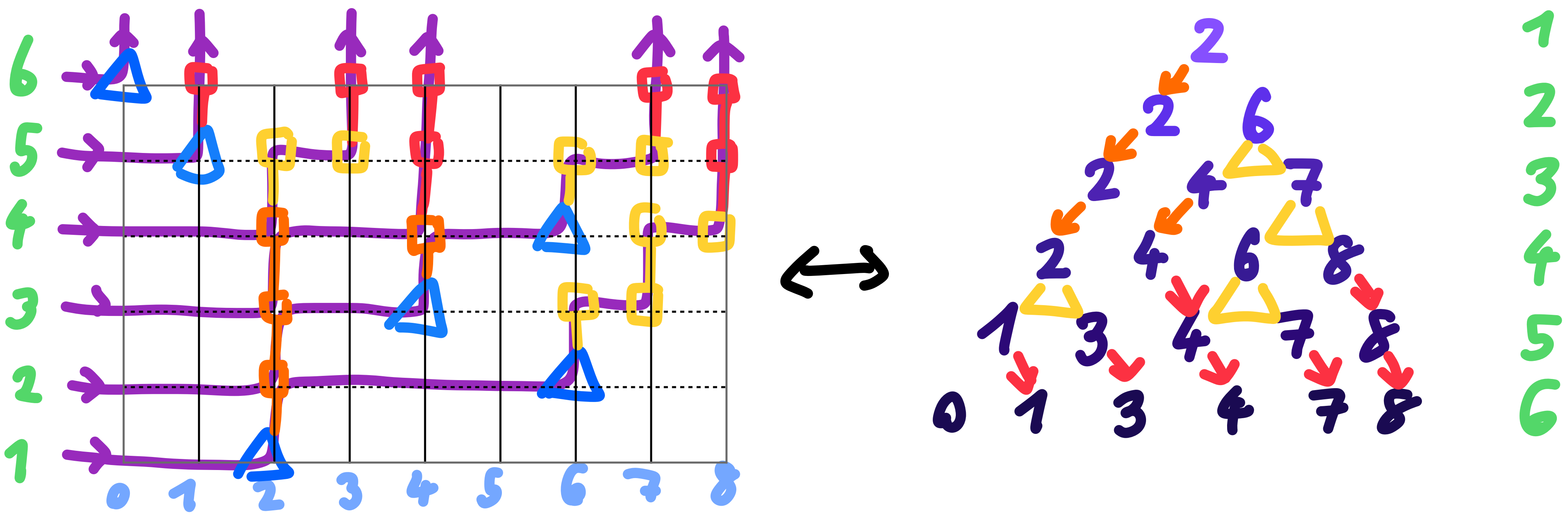
$\boxed{\text{Assume } \lambda_1 > \lambda_2 > \dots > \lambda_n \geq 0.}$ Set $s = q^{-\frac{1}{2}}$.



Prefactor swallows weight of one $\rightarrow\!\uparrow$ - configuration per row.
Say the left most such vertex of each row. Note that then all horizontal edges to the left in that row are part of a path and have been handled already.

Bijection: higher spin 6VM ↔ DAMTs

$\boxed{\text{Assume } \lambda_1 > \lambda_2 > \dots\dots > \lambda_n \geq 0.}$ Set $s = q^{-\frac{1}{2}}$.



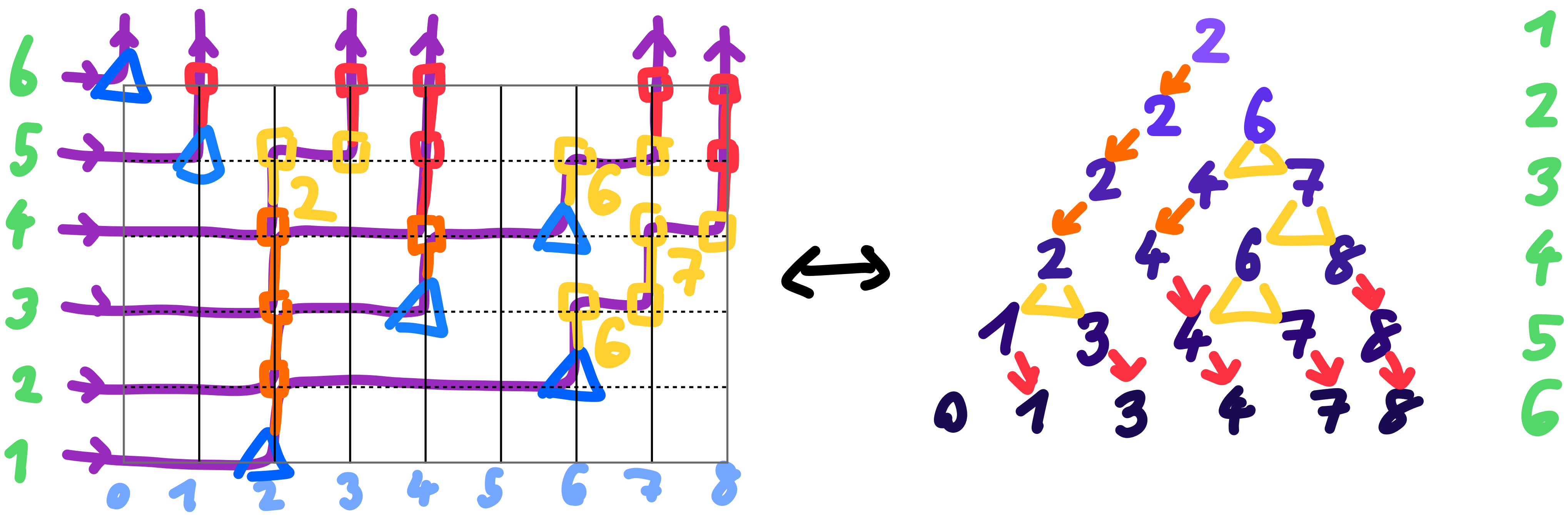
Claim: Pairs of & -vertices

↔ $\begin{smallmatrix} & a & \\ b & & c \end{smallmatrix}$ -configurations

Proof: Appear in pairs, since any path that enters a row via has to leave it again via . (Maybe some & in between)

Bijection: higher spin 6VM $\leftrightarrow$ DAMTs

$\boxed{\text{Assume } \lambda_1 > \lambda_2 > \ldots > \lambda_n \geq 0.}$ Set $s = q^{-\frac{1}{2}}$.



Vertical edge of $\bullet\!\!\rightarrow$ gives special entry, since clearly no vertical edge directly under $\bullet$.
And most importantly:

$$w(\text{—}\!\!\bullet\,)\,w(\,\ulcorner\!\!\text{—}\,) = w(\searrow) + w(\swarrow) + w(\downarrow)$$


Thank you
very much
for your
attention!
Have a
nice day!