Universal Properties of Systems of Positive Linear Catalytic Equations

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Joint work with Cyril Banderier:
arXiv:2605.22241

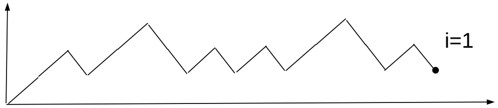
Lattice Path Conference 2026, TU Wien, July 20–24, 2026

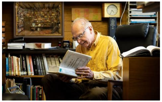
Positive polynomial equations: universal asymptotics

Many combinatorial problems are encoded by equations like $F(z) = Q(z, F(z))$ and catalytic equations like $F(z,u) = zQ\left(F(z,u), \frac{F(z,u)-F(z,0)}{u}, z, u\right)$.

They always have the same type of Puiseux expansions, and therefore the same universal asymptotics.

Linear equation $\rightsquigarrow$ polar singularity	Linear catalytic equation $\rightsquigarrow$ square-root singularity
Nonlinear equation $\rightsquigarrow$ square-root singularity	Nonlinear catalytic equation $\rightsquigarrow$ $3/2$-singularity





Knuth, 1969

$f_{n,i} :=$ number of non-negative paths from $(0,0) \to (n,i)$

$$F_i(z) = \sum_{n\ge 0} f_{n,i} z^i \qquad F(z,u) = \sum_{i\ge 0} F_i(z) u^i = \sum_{n,i\ge 0} f_{n,i} z^n u^i$$

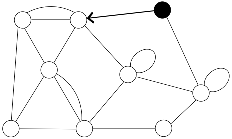
u is a catalytic variable which allows us to get an equation:

$$\boxed{F(z,u) = 1 + zuF(z,u) + z\frac{F(z,u)-F(z,0)}{u}}$$

$$F(z,0) = F_0(z) = \frac{1-\sqrt{1-4z^2}}{2z^2} \quad \text{(square-root singularity)}$$

$$f_{2n,0} = [z^{2n}]F(z,0) = \frac{1}{n+1}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi}}\, n^{-3/2}\, 4^n$$

One quadratic catalytic equation: planar map example



Tutte, 1962

$m_{n,k}$= number of planar maps with n edges and outer face valency k

$$M(z,u) = \sum_{n,k} m_{n,k} z^n u^k$$

u is a catalytic variable leading to the functional equation:

$$\boxed{M(z,u) = 1 + zu^2 M(z,u)^2 + uz \frac{uM(z,u) - M(z,1)}{u-1}}$$

$$M(z,1) = -\frac{1}{54z^2}\left(1 - 18z - (1-12z)^{3/2}\right) \quad (\mathbf{3/2\text{-singularity}})$$

$$m_n = [z^n]M(z,1) = \frac{2(2n)!}{(n+2)!n!}3^n \sim \frac{2}{\sqrt{\pi}} \cdot n^{-5/2} 12^n$$

One positive linear equation

Theorem (Polar singularity)

For $Q_0(z)$, $Q_1(z)$ polynomials with non-negative coefficients, the linear equation

$$\boxed{y(z) = Q_0(z) + zQ_1(z)y(z)}$$

$\Longrightarrow$ *$y(z)$ has a dominating polar singularity and*

$$y_n = [z^n]y(z) \sim c \cdot z_0^{-n},$$

(the constants c depends on $n \bmod m$ if one has m dominant singularities). $z_0 > 0$ is given by $z_0 Q_1(z_0) = 1$.

Proof: simple analysis of $y(z) = Q_0(z)/(1 - zQ_1(z))$.
Typical examples: Fibonacci numbers, unconstrained lattice paths, regular expressions.

One positive nonlinear equation

Theorem (Bender 1974, Canfield, Meir–Moon 1990)

For $Q(z,y)$ polynomial with non-negative coefficients (with $Q(0,0)=0$ and $Q_{yy}\ne 0$, so it is nonlinear),

$$\boxed{\boxed{y(z) = Q(z, y(z))}}$$

$\Longrightarrow$ *$y(z)$ has a dominating square-root singularity and*

$$y_n = [z^n]y(z) \sim c \cdot n^{-3/2} z_0^{-n}$$

(the constant c depends on $n \bmod m$ if one has m dominant singularities). $z_0 > 0$ satisfies $y_0 = Q(z_0, y_0)$ and $1 = Q_y(z_0, y_0)$ for some $y_0 > 0$.

Proof. Based on the analysis of the singular point (z_0, y_0) of the curve $y = Q(z,y)$ that leads to $y(z) = g(z) - h(z)\sqrt{1 - z/z_0}$.

One positive linear catalytic equation

Theorem (Drmota–Noy–Yu 2022)

For $Q_i(z,u)$ polynomials with non-negative coefficients such that $Q_{1,u} \neq 0$ and $u \not| Q_2$,

$$\boxed{\boxed{F(z,u) = Q_0(z,u) + zF(z,u)Q_1(z,u) + z\frac{F(z,u) - F(z,0)}{u}Q_2(z,u)}}$$

$\Longrightarrow$ *$F(z,0)$ has a dominating square-root singularity and*

$$f_n = [z^n]F(z,0) \sim c \cdot n^{-3/2} z_0^{-n}$$

(the constant c depends on n mod m if one has m dominant singularities).

[Drmota–Hainzl 2023]: same claim if **second differences** are involved.

One positive nonlinear catalytic equation

Theorem (Drmota–Noy–Yu 2022, 3/2-singularity)

For $Q(y_0, y_1, z, u)$ polynomial with non-negative coefficients that is nonlinear in y_0, y_1 (and depends on y_0, y_1),

$$\boxed{\boxed{F(z,u) = zQ\left(F(z,u), \frac{F(z,u) - F(z,0)}{u}, z, u\right)}}$$

$\Longrightarrow$ *has a dominating 3/2-singularity and*

$$f_n = [z^n]\, F(z,0) \sim c \cdot n^{-5/2} z_0^{-n}$$

(the constant c depends on $n \bmod m$ if one has m dominant singularities).

[Drmota–Hainzl 2023]: same claim if **second differences** are involved.

Strongly connected systems of positive polynomial equations

Some combinatorial problems are encoded by a system of equations

$$\left\{ y_j(z) = Q_j(z, \mathbf{y}(z)) \right.$$

or by a system of catalytic equations

$$\left\{ F_j(z,u) = zQ_j\left(\mathbf{F}(z,u), \frac{\mathbf{F}(z,u) - \mathbf{F}(z,0)}{u}, z, u\right) \right. .$$

Linear system of equations $\rightsquigarrow$ polar singularity	Linear system of catalytic equations $\rightsquigarrow$ square-root singularity: THIS TALK
Nonlinear system of equations $\rightsquigarrow$ square-root singularity	Nonlinear system of catalytic equations $\rightsquigarrow$???

Conjecture. Positive strongly connected nonlinear systems of catalytic equations have always a dominant $3/2$-singularity.

Remark. There is a recent result for **one equation** by Alice Contat and Nicolas Curien that supports this conjecture.

System of functional equations

$Q_1, \ldots Q_d$ polynomials with non-negative coefficients.
$y_1, \ldots, y_d$ solutions of the system:

$$\begin{cases} y_1 = Q_1(z, y_1, \ldots, y_d), \\ \vdots \\ y_d = Q_d(z, y_1, \ldots, y_d). \end{cases}$$

The dependency graph of such a system is a graph on the vertices $\{1, \ldots d\}$ with directed edges from $i \to j$ if Q_j depends on y_i.
Such a system is called strongly connected if the dependency graph is strongly connected.
Morally, no subsystem can be solved before solving the whole system!

Systems of functional equations (strongly connected case)

Theorem (Drmota, Lalley, Woods ∼1997)

$\boxed{\boxed{\mathbf{y} = \mathbf{Q}(z, \mathbf{y})}}$ *non-negative (and well defined) polynomial system of* $d \geq 1$ *equations with a strongly connected dependency graph.*

Then the situation is the same as for a single equation.

If the system is linear, then we have a common polar singularity and

$$[z^n]y_1(z) \sim c \cdot z_0^{-n},$$

whereas if it is nonlinear, then we have a square-root singularity and

$$[z^n]y_1(z) \sim c \cdot n^{-3/2} z_0^{-n}$$

(the constant c *depends on* $n \bmod m$ *if one has* m *dominant singularities).*

Systems of functional equations (not strongly connected)

Theorem (Banderier–Drmota 2014)

$\boxed{\boxed{\mathbf{y} = \mathbf{Q}(z, \mathbf{y})}}$ *non-negative (and well defined) polynomial system of equations.*

$\Longrightarrow$

$$[z^n]\, y_1(z) \sim c\, n^{-1-\alpha}\, \rho^{-n}$$

(the constants c, α, ρ depend on $n \bmod m$ if one has m dominant singularities), where the critical exponent α is a dyadic number:

$$\alpha \in \{2^{-\kappa} : \kappa \ge 1\} \cup \{-m2^{-\kappa} : m \ge 1, \kappa \ge 0\}.$$

Remark. κ is the number of "chained" strongly connected components; for $\kappa = 1$, this gives $\alpha = 1/2$, a square-root singularity.

Context-free grammars

The solutions of a **positive polynomial** system

$$\begin{cases} y_1 &= Q_1(z, y_1, \ldots, y_d), \\ & \vdots \\ y_d &= Q_d(z, y_1, \ldots, y_d). \end{cases}$$

are generating functions of a finite context-free grammar.
Therefore, we say that a (generating) function **corresponds to a finite grammar** if it is one of the solution functions of a **finite positive polynomial system of equations**.

Systems of linear catalytic equations

$P_i(x,u)$, $Q_{i,j,\ell}(x,u)$ polynomials with non-negative coefficients

$$\boxed{F_i(x,u) = P_i(x,u) + x\sum_{j=1}^{d}\sum_{\ell=0}^{L} Q_{i,j,\ell}(x,u)\Delta^\ell F_j(x,u) \qquad (\text{for } 1\le i\le d)}$$

with $\Delta G(x,u) = \dfrac{G(x,u)-G(x,0)}{u}$ and $\Delta^\ell G(x,u) = \Delta\left(\Delta^{\ell-1}G(x,u)\right)$.

This is direct relation to an infinite system of equations for the functions $F_{i,k}(x) = [u^k]F_i(x,u)$:

$$\boxed{F_{i,k}(x) = P_{i,k}(x) + x\sum_{j=1}^{d}\sum_{\ell=0}^{L}\sum_{m=0}^{\min(J,k)} Q_{i,j,\ell,m}(x)F_{j;k+\ell-m,0}(x)}$$

$(1\le i\le d,\ k\ge 0,\ J = \max_{i,j,\ell}\deg_u Q_{i,j,\ell}(x,u))$

We say that the catalytic system is strongly connected if the corresponding infinite system is strongly connected.

Results

Consider $F_i(x,u) = \sum_{k\ge 0} F_{i;k}(x)u^k$ (for $1 \le i \le d$) solutions of a positive linear system of polynomial catalytic equations.

$$\left\{ F_i(x,u) = P_i(x,u) + x \sum_{j=1}^{d} \sum_{\ell=0}^{L} Q_{i,j,\ell}(x,u)\Delta^\ell F_j(x,u) \qquad (i = 1,\ldots,d). \right.$$

Theorem

Each function $F_{i;k}(x)$ corresponds to a finite grammar.

Theorem

If the initial system is strongly connected, then all solution functions $F_{i;k}(x)$ have a common dominating square-root singularity.

Proof strategy

- Translation into a lattice path counting problem with weights.
- Apply prime walk decomposition
- Translation into an infinite nonlinear system (without a catalytic variable)
- The solution functions of a finite subsystem determine everything.
- Special care and another decomposition in the strongly connected case.

Lattice path counting problem (setup)

One equation with $P(x,u) = 1$. $F(x,u) = \sum_{k\ge 0} F_{k,0}(x) u^k$:

$$\boxed{F(x,u) = 1 + x \sum_{\ell=0}^{L} \sum_{j=0}^{J} Q_{\ell,j}(x) u^j \Delta^\ell F(x,u)}$$

$$\boxed{F_{k,0}(x) = \delta_{k,0} + x \sum_{\ell=0}^{L} \sum_{j=0}^{\min(k,J)} Q_{\ell,j}(x) F_{\ell+k-j,0}(x)}$$

Step sets at height k: an ultimately constant sequence.

$$\mathcal{S}_k = \bigcup_{\ell=0}^{L} \bigcup_{j=0}^{\min(k,J)} \{(1+r, \ell - j) : 0 \le r \le \deg Q_{\ell,j}, [x^r] Q_{\ell,j}(x) \ne 0\}$$

$$\boxed{\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_J = \mathcal{S}_{J+1} = \ldots}$$

Lattice path counting problem (example)

$$F(x,u) = 1 + xuF(x,u) + x(1+u^3)\Delta^1 F(x,u)$$

$$\begin{cases} F_{0,0}(x) & = 1 + xF_{1,0}(x), \\ F_{1,0}(x) & = xF_{0,0}(x) + xF_{2,0}(x), \\ F_{2,0}(x) & = xF_{1,0}(x) + xF_{3,0}(x), \\ F_{3,0}(x) & = xF_{2,0}(x) + xF_{4,0}(x) + xF_{1,0}(x), \\ F_{4,0}(x) & = xF_{3,0}(x) + xF_5(x) + xF_{2,0}(x), \\ & \vdots \end{cases}$$

The step sets $\mathcal{S}_k$ at height k are:

$$\begin{cases} \mathcal{S}_0 & = \{(1,1)\}, \\ \mathcal{S}_1 & = \{(1,1),(1,-1)\}, \\ \mathcal{S}_2 & = \{(1,1),(1,-1)\}, \\ \mathcal{S}_k & = \{(1,1),(1,-1),(1,-2)\} \qquad (k \ge 3). \end{cases}$$

Prime walks

Prime walks are the building block of any path! [Banderier–Lackner–Wallner 2020] generalizing/simplifying [Labelle–Yeh 1990].

N.B.: Our model is more complex (At height k, all our paths have steps from the set $\mathcal{S}_k$); this entails heavier notations.

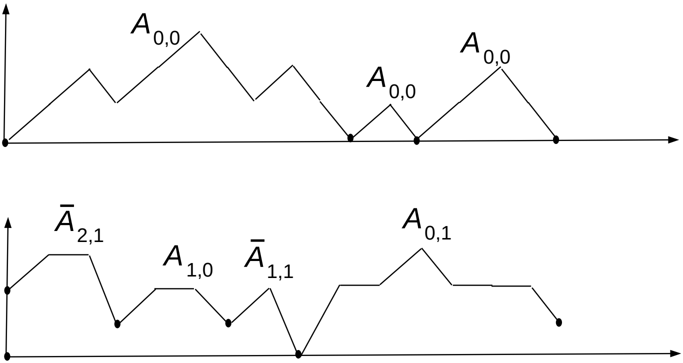
A prime walk $\mathcal{A}_{k,i}$ is a path that starts at level k, ends at level $k+i$, and stays strictly above the level $k+i$ in between.

A reverse prime walk $\overline{\mathcal{A}}_{k,i}$ is a path that starts at level k, ends at level $k-i$, and stays strictly above the the level k in between.

Generating functions:

$$A_{k,i}(x) \qquad\qquad \overline{A}_{k,i}(x)$$

Example of prime walk decompositions



Gen. functions for prime walk decomposition

$F^{\ge k}_{k_1,k_2}(x)$: generating function of walks that connect level k_1 with level k_2 and stay above level k:

$$F^{\ge k}_{k_1,k_1}(x) = 1 + \sum_{m=k}^{k_1} \overline{A}_{k_1,k_1-m}(x) F^{\ge k}_{m,k_1}(x) \qquad (k \le k_1),$$

$$F^{\ge k}_{k_1,k_2}(x) = \sum_{m=k}^{k_1} \overline{A}_{k_1,k_1-m}(x) F^{\ge k}_{m,k_2}(x) + \sum_{m=k_1+1}^{k_2} A_{k_1,m-k_1}(x) F^{\ge m}_{m,k_2}(x) \ (k \le k_1 < k_2)$$

$$F^{\ge k}_{k_1,k_2}(x) = \sum_{m=k}^{k_1} \overline{A}_{k_1,k_1-m}(x) F^{\ge k}_{m,k_2}(x) \qquad (k \le k_2 < k_1).$$

Recall that

$$F(x,u) = \sum_{k\ge 0} F_{k,0}(x) u^k \quad \text{with} \quad F_{k,0}(x) = F^{\ge 0}_{k,0}(x).$$

Gen. functions for prime walk decomposition

$$
\begin{aligned}
A_{k,i}(x) = &\sum_{0\le \ell\le L, 0\le j\le k\,:\,\ell-j=i} xQ_{\ell,j}(x)\\
&+ \sum_{i_1>i,\, i_2>i}\ \sum_{0\le \ell_1\le L, 0\le j_1\le k\,:\,\ell_1-j_1=i_1} xQ_{\ell_1,j_1}(x)\ F^{\ge\, k+i+1}_{k+i_1,k+i_2}(x)\\
&\qquad \sum_{0\le \ell_2\le L, 0\le j_2\le \min(k+i_2,J)\,:\,\ell_2-j_2=-(i_2-i)} xQ_{\ell_2,j_2}(x)\\
\overline{A}_{k,i}(x) = &\sum_{0\le \ell\le L, 0\le j\le k\,:\,\ell-j=-i} xQ_{\ell,j}(x)\\
&+ \sum_{i_1>0,\, i_2>0}\ \sum_{0\le \ell_1\le L, 0\le j_1\le k\,:\,\ell_1-j_1=i_1} xQ_{\ell_1,j_1}(x)\ F^{\ge\, k+1}_{k+i_1,k+i_2}(x)\\
&\qquad \sum_{0\le \ell_2\le L, 0\le j_2\le \min(k+i_2,J)\,:\,\ell_2-j_2=-(i+i_2)} xQ_{\ell_2,j_2}(x)
\end{aligned}
$$

$$A_{k,i}(x) = A_{J,i}(x), \qquad \overline{A}_{k,i}(x) = \overline{A}_{J,i}(x) \quad \text{(for } k\ge J\text{, and } = 0 \text{ otherwise).}$$

Gen. functions for prime walk decomposition

Lemma

The finite system of equations for the functions

$$
\begin{aligned}
F^{\geq k}_{k_1,k_2}(x), &\quad 0 \le k_1 < L + J,\ 0 \le k_2 < 2J,\ 0 \le k \le \min(J, k_1, k_2)\\
A_{k,i}(x), &\quad 0 \le k \le J,\ 0 \le i < L\\
\overline{A}_{k,i}(x), &\quad 0 \le k \le J,\ 0 \le i < J,
\end{aligned}
$$

is a positive polynomial system of equations that determines uniquely all these functions.
Thus, all functions $F^{\geq k}_{k_1,k_2}(x)$ correspond to a finite grammar.

Gen. functions for prime walk decomposition (example)

$$\boxed{F(x,u) = 1 + xuF(x,u) + x(1+u^3)\Delta F(x,u)}$$

From the step set structure it follows that:

$$F^{\geq k}_{k_1,k_2}(x) = F^{\geq 1}_{k_1-k+1,k_2-k+1} \qquad (k \geq 1).$$

The prime walk generating functions satisfy:

$$A_{0,0}(x) = xF^{\geq 1}_{1,1}(x)x, \qquad A_{0,1}(x) = x, \qquad A_{1,i}(x) = 0 \quad (i \geq 2)$$

and

$$\begin{aligned} A_{k,0}(x) &= xF^{\geq k+1}_{k+1,k+1}(x)x + xF^{\geq k+1}_{k+1,k+2}(x)x \\ &= xF^{\geq 1}_{1,1}(x)x + xF^{\geq 1}_{1,2}(x)x = A_{1,0}(x) \quad (k \geq 1), \\ A_{k,1}(x) &= x \quad (k \geq 1), \qquad A_{k,i}(x) = 0 \quad (k \geq i \geq 2). \end{aligned}$$

Gen. functions for prime walk decomposition (example)

The reverse prime walk generating functions satisfy:

$$
\begin{aligned}
\overline{A}_{k,0}(x) &= A_{k,0}(x) \quad (k \ge 0), \\
\overline{A}_{1,1}(x) &= x, \\
\overline{A}_{k,1}(x) &= x + xF^{\ge k+1}_{k+1,k+1}(x)x \\
&= x + xF^{\ge 1}_{1,1}(x)x = \overline{A}_{2,1} \quad (k \ge 2), \\
\overline{A}_{k,2}(x) &= \overline{A}_{3,2}(x) = x \quad (k \ge 3), \\
\overline{A}_{k,i}(x) &= 0 \quad (k \ge i \ge 3).
\end{aligned}
$$

The only unknown prime walk functions are $A_{0,0}(x),\ A_{1,0}(x),\ \overline{A}_{2,1}(x)$.
Furthermore we have

$$
\begin{aligned}
F^{\ge 1}_{1,1}(x) &= 1 + \overline{A}_{1,0}(x)F^{\ge 1}_{1,1}(x) \\
&= 1 + A_{1,0}(x)F^{\ge 1}_{1,1}(x), \\
F^{\ge 1}_{1,2}(x) &= \overline{A}_{1,0}(x)F^{\ge 1}_{1,2}(x) + A_{1,1}(x)F^{\ge 2}_{2,2}(x) \\
&= A_{1,0}(x)F^{\ge 1}_{1,2}(x) + xF^{\ge 1}_{1,1}(x).
\end{aligned}
$$

Gen. functions for prime walk decomposition (example)

This leads to a positive strongly connected system (and thus to our beloved square-root Puiseux behavior!):

$$\begin{cases} A_{1,0}(x) & = xF_{1,1}^{\geq 1}(x)x + xF_{1,2}^{\geq 1}(x)x, \\ F_{1,1}^{\geq 1}(x) & = 1 + A_{1,0}(x)F_{1,1}^{\geq 1}(x), \\ F_{1,2}^{\geq 1}(x) & = A_{1,0}(x)F_{1,2}^{\geq 1}(x) + xF_{1,1}^{\geq 1}(x). \end{cases}$$

With the help of these three functions, all other functions can be determined. For example

$$A_{0,0}(x) = xF_{1,1}^{\geq 1}(x)x, \qquad F_{0,0}^{\geq 0}(x) = \frac{1}{1 - A_{0,0}(x)}.$$

Conclusion

- Universal critical exponent α for $f_n \sim cn^\alpha \rho^{-n}$ in linear positive systems of catalytic equations.
- $\alpha = -3/2$ if strongly connected system, and dyadic if not.
- Proof unexpectedly uses the combinatorics of prime walks and context-free grammars
- Perspectives: Gaussian law, $\alpha = -5/2$ for nonlinear case.

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- Universal critical exponent α for $f_n \sim cn^\alpha \rho^{-n}$ in linear positive systems of catalytic equations.
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Thank You!