Partition identities, particle motion and lattice paths

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Joint work with Frédéric Jouhet and Isaac Konan (arXiv:2403.05414), Jihyeug Jang and Frédéric Jouhet (arXiv:2507.13239), and Jihyeug Jang (arXiv:2603.05300).

Outline

1. Introduction
2. Warnaar's particle motion
3. Generalisation
4. Applications

Integer partitions

Definition

A partition of a non-negative integer n is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

- The integer $n = |\lambda|$ is the weight of λ.
- The integers λ_k are the parts of λ.
- The integer ℓ is the length of λ.

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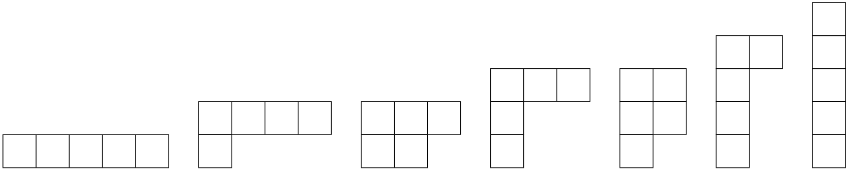
Example: Partitions of 5:

$$(5) \quad (4,1) \quad (3,2) \quad (3,1,1) \quad (2,2,1) \quad (2,1,1,1) \quad (1,1,1,1,1)$$

Young diagrams:



Generating functions

Notation :

$$(a)_n = (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \text{ for } n \in \mathbb{N} \cup \{\infty\},$$

$$(a_1,\ldots,a_m)_k = (a_1,\ldots,a_m;q)_k := (a_1)_k \cdots (a_m)_k.$$

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If $P_{k,N}(n)$ is the number of partitions of n into parts congruent to $k \mod N$:

$$\sum_{n\geq 0} P_{k,N}(n)q^n = \frac{1}{(q^k;q^N)_\infty}.$$

If $p_k(n)$ is the number of partitions of n into parts at most k (or equivalently, at most k parts):

$$\sum_{n\geq 0} p_k(n)q^n = \frac{1}{(q;q)_k}.$$

The Rogers–Ramanujan identities

Rogers 1894, Ramanujan 1914, Rogers–Ramanujan 1919

Let $a \in \{0, 1\}$. Then

$$\sum_{n \geq 0} \frac{q^{n^2+(1-a)n}}{(q)_n} = \frac{1}{(q^{2-a}, q^{3+a}; q^5)_\infty}.$$

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Combinatorial version (MacMahon 1916)

Let $a \in \{0,1\}$. The number of partitions λ of n with difference condition $\lambda_i - \lambda_{i+1} \geq 2$ for all i, s.t. the part 1 appears at most a times, is equal to the number of partitions of n into parts congruent to $\pm(2-a)$ modulo 5.

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Appear in combinatorics (Andrews, Bressoud, Warnaar,...), statistical mechanics (Andrews, Baxter,...), number theory (Ono, Zagier,...), representation theory (Lepowski, Milne, Wilson,...), algebraic geometry (Mourtada,...),...

The Andrews–Gordon identities

Theorem (Gordon 1961)

Let $k \geq 1$ and $0 \leq r \leq k$.
Let $\mathcal{T}_{r,k}(n)$ be the set of partitions λ of n with difference condition $\lambda_i - \lambda_{i+k} \geq 2$ for all i, s.t. the part 1 appears at most $(k-r)$ times.
Let $\mathcal{E}_{r,k}(n)$ be the set of partitions of n into parts ***not*** *congruent to $0, \pm(k-r+1)$ modulo $2k+3$.*
Then for all n, $|\mathcal{T}_{r,k}(n)| = |\mathcal{E}_{r,k}(n)|$.

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Theorem (Andrews–Gordon identities (Andrews 1974))

Let $k \geq 1$ and $0 \leq r \leq k$ be two integers. We have

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\ldots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \frac{(q^{2k+3}, q^{k-r+1}, q^{k+r+2}; q^{2k+3})_\infty}{(q)_\infty}.$$

The Rogers–Ramanujan identities correspond to the case $k=1$.

Combinatorial interpretation

For $n \geq 0$, the q-binomial coefficient is defined as:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_k (q)_{n-k}}.$$

Note that $\left[{n \atop k} \right]_q = 0$ if $k < 0$ or $k > n$.

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The sum-side of the Andrews–Gordon identities can be rewritten as:

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2 + \cdots + s_k^2 + s_{k-r+1} + \cdots + s_k}}{(q)_{s_1 - s_2} \cdots (q)_{s_{k-1} - s_k} (q)_{s_k}} =$$

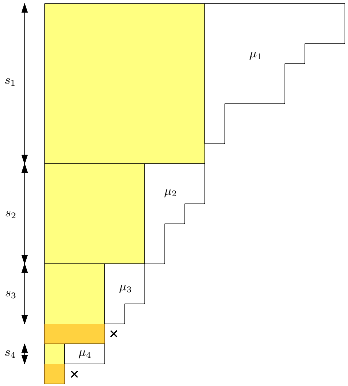
$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2 + \cdots + s_k^2 + s_{k-r+1} + \cdots + s_k}}{(q)_{s_1}} \begin{bmatrix} s_1 \\ s_1 - s_2 \end{bmatrix}_q \cdots \begin{bmatrix} s_{k-1} \\ s_{k-1} - s_k \end{bmatrix}_q.$$

Combinatorial interpretation

Using the expression

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1}} \begin{bmatrix} s_1 \\ s_1-s_2 \end{bmatrix}_q \cdots \begin{bmatrix} s_{k-1} \\ s_{k-1}-s_k \end{bmatrix}_q,$$

there is a natural combinatorial interpretation of the sum side in terms of Durfee squares (Andrews):



Combinatorial interpretation

But it is not obvious combinatorially that

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}}$$

is the generating functions for partitions λ in $\mathcal{T}_{r,k}$, i.e. with difference condition $\lambda_i - \lambda_{i+k} \geq 2$ for all i, s.t. the part 1 appears at most $(k-r)$ times.

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Warnaar (1997) did it bijectively via *particle motion*.

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2 Warnaar's particle motion

3 Generalisation

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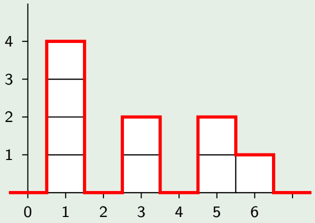
Frequency sequences and lattice paths

Definition

Let λ be a partition. The *frequency sequence* of λ is the infinite sequence $(f_i)_{i\geq 1}$ such that for all $i \geq 1$, f_i is the number of appearances of the part i in λ.

Graphical representation: lattice paths, or equivalently piles of boxes

The frequency sequence of $(6,5,5,3,3,1,1,1,1)$ is $(4,0,2,0,2,1,0,\dots)$.



Reformulation in terms of frequency sequences

Theorem (Gordon 1961)

Let $k \geq 1$ and $0 \leq r \leq k$.
Let $\mathcal{T}_{r,k}(n)$ be the set of partitions λ of n with difference condition $\lambda_i - \lambda_{i+k} \geq 2$ for all i, s.t. the part 1 appears at most $(k-r)$ times.
Equivalently, $\mathcal{T}_{r,k}(n)$ is the set of partitions λ of n whose frequency sequence $(f_i)_{i \geq 1}$ satisfies

$$f_i + f_{i+1} \leq k \text{ for all } i,$$
$$f_1 \leq k - r.$$

*Let $\mathcal{E}_{r,k}(n)$ be the set of partitions of n into parts **not** congruent to $0, \pm(k-r+1)$ modulo $2k+3$.*
Then for all n, $|\mathcal{T}_{r,k}(n)| = |\mathcal{E}_{r,k}(n)|$.

Warnaar's idea

In *The Andrews-Gordon identities and q-multinomial coefficients*, Comm. Math. Phys. (1997), Warnaar proved a finite version of the Andrews–Gordon identities. Let us decribe the sum side of this identity.

Let $\mathcal{T}_{r,k,L}$ be the set of partitions λ with frequency sequence $(f_i)_{i\geq 1}$ s.t.

$$\begin{aligned} f_i + f_{i+1} &\leq k \text{ for all } i, \\ f_1 &\leq k-r, \\ f_i &= 0 \text{ for all } i \geq L. \end{aligned}$$

Then

$$\sum_{\lambda \in \mathcal{T}_{r,k,L}} q^{|\lambda|} = \sum_{s_1 \geq \cdots \geq s_k \geq 0} q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k} \prod_{j=1}^{k} \begin{bmatrix} s_j - s_{j+1} + m_j \\ s_j - s_{j+1} \end{bmatrix}_q,$$

where the m_j's are related to Lie algebras and tend to ∞ when $L \to \infty$.

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Idea: any partition in $\mathcal{T}_{r,k,L}$ can be obtained in a unique way from a minimal partition with weight $s_1^2 + \cdots + s_k^2 + s_{k-r+1} + \cdots + s_k$ by performing a certain number of *particle motions*.

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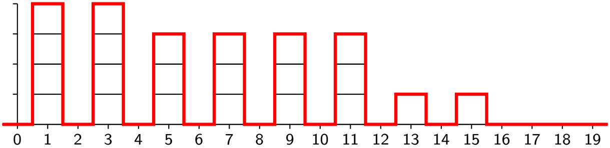
For $r = 0$, the minimal partitions are of the shape

$$(\underbrace{f_1 = k, f_2 = 0, \ldots, k, 0}_{s_k \text{ pairs}}, \ldots, \underbrace{i, 0, \ldots, i, 0}_{s_i - s_{i+1} \text{ pairs}}, \ldots, \underbrace{1, 0, \ldots, 1, 0}_{s_1 - s_2 \text{ pairs}}, 0, \ldots).$$

Example: $k = 4, s_1 = 8, s_2 = s_3 = 6, s_4 = 2$



Particle motion starting at index u

Let f be a frequency sequence.
Let u be an index such that, letting $h := f_u + f_{u+1}$,

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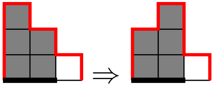
Repeat the following process

- If $f_{u+1} + f_{u+2} < h$, then transform (f_u, f_{u+1}) to $(f_u - 1, f_{u+1} + 1)$, leave u unchanged (**particle motion**).



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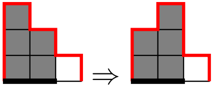
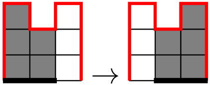
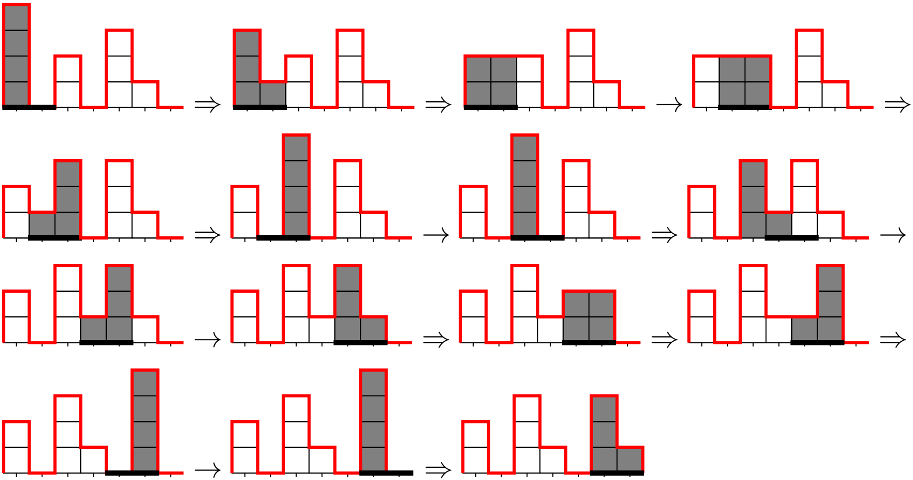


- If $f_{u+1} + f_{u+2} = h$, then leave (f_u, f_{u+1}) unchanged, but change u to $u + 1$ (**focus shift**).



Example of particle motion



Application of 9 particle motions starting from index 1 in the frequency sequence $f = (4, 0, 2, 0, 3, 1, 0, 0, \cdots)$.

Properties of particle motion

Let $f = (f_0, f_1, \dots)$ be a frequency sequence and m be a non-negative integer. Let u be a non-negative integer such that $f_u + f_{u+1} = h$ and $f_i + f_{i+1} \leq h$ for all $i \geq u$.
We denote by $\mathrm{pm}_u^{(m)}(f)$ the frequency sequence obtained by applying applying m particle motions starting at index u in f.

Properties

- For all m, $|\mathrm{pm}_u^{(m)}(f)| = |f| + m$.
- If f satisfies the frequency conditions of Andrews–Gordon, so does $\mathrm{pm}_u^{(m)}(f)$ for any m.

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In Warnaar's formula: $q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}$ comes from the minimal partition described earlier, and $\prod_{j=1}^{k} \begin{bmatrix} s_j - s_{j+1} + m_j \\ s_j - s_{j+1} \end{bmatrix}_q$ comes from the particle motions.

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A generalised bijection (D.–Jouhet–Konan 2024, D.–Jang–Jouhet 2025)

Main (simple) idea: allow parts of size 0.

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From now on, a partition of a non-negative integer n is a finite non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

The frequency sequence of λ is the infinite sequence $(f_i)_{i \geq 0}$ such that for all $i \geq 0$, f_i is the number of appearances of the part i in λ.

The parts equal to 0 contribute to the length of the partitions, but not to their weight.

Frame sequences

Definition

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ be a k-tuple of partitions, each of which may be empty or contain parts 0. The *weight* $|\boldsymbol{\lambda}|$ *of* $\boldsymbol{\lambda}$ *is defined by* $\sum_i |\lambda^{(i)}|$, *and the length* $\ell(\boldsymbol{\lambda})$ *of* $\boldsymbol{\lambda}$ *is defined by* $\sum_i \ell(\lambda^{(i)})$. *Let* $s_1, \ldots, s_k$ *s.t.* $s_1 \geq \cdots \geq s_k \geq 0$ *and* $\ell(\lambda^{(i)}) = s_i - s_{i+1}$ *for all* i, *where* $s_{k+1} := 0$.
The frame sequence associated to $\boldsymbol{\lambda}$ *is the frequency sequence*

$$\begin{aligned}\mathrm{fs}(\boldsymbol{\lambda}) &= \mathrm{fs}(s_1, \ldots, s_k)\\ &:= (\underbrace{f_0 = k, f_1 = 0, \ldots, k, 0}_{s_k \text{ pairs}}, \ldots, \underbrace{i, 0, \ldots, i, 0}_{s_i - s_{i+1} \text{ pairs}}, \ldots, \underbrace{1, 0, \ldots, 1, 0}_{s_1 - s_2 \text{ pairs}}, 0, \ldots).\end{aligned}$$

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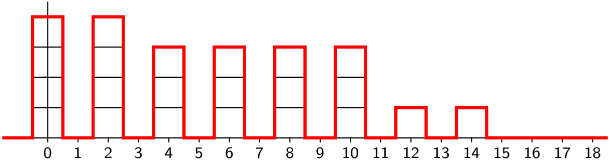
Ex: $\boldsymbol{\lambda} = ((3,1), \emptyset, (6,6,5,3), (19,0)) \Rightarrow k = 4, s_1 = 8, s_2 = s_3 = 6, s_4 = 2$



The bijection

Definition

Let k be a non-negative integer. Define

$$\mathcal{P}_k := \{(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})) : \boldsymbol{\lambda} \text{ is a } k\text{-tuple of partitions}\},$$
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Theorem (D.–Jouhet–Konan 2024, reformulation D.–Jang–Jouhet 2025)

Let $\Lambda : \mathcal{P}_k \to \mathcal{A}_k$ be defined by

$$\Lambda(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})) = \left(\mathrm{pm}_0^{(\lambda_0)} \circ \mathrm{pm}_2^{(\lambda_1)} \circ \cdots \circ \mathrm{pm}_{2(s_1-2)}^{(\lambda_{s_1-2})} \circ \mathrm{pm}_{2(s_1-1)}^{(\lambda_{s_1-1})}\right)(\mathrm{fs}(\boldsymbol{\lambda})),$$

where $(\lambda_{s_1-1}, \lambda_{s_1-2}, \ldots, \lambda_1, \lambda_0) = (\lambda_1^{(1)}, \ldots, \lambda_{s_1-s_2}^{(1)}, \ldots, \lambda_1^{(k)}, \ldots, \lambda_{s_k}^{(k)})$.
Then Λ is a weight- and length-preserving bijection between $\mathcal{P}_k$ and $\mathcal{A}_k$.

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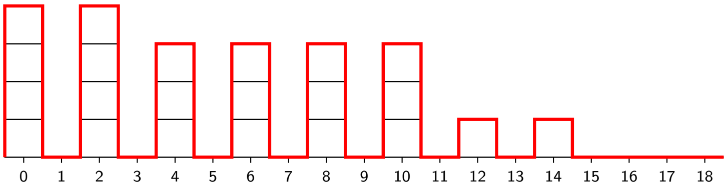
Then Λ *is a weight- and length-preserving bijection between* $\mathcal{P}_k$ *and* $\mathcal{A}_k$.

This bijection induces many bijections between subsets of $\mathcal{P}_k$ and $\mathcal{A}_k$.

$\Lambda(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$ where $\boldsymbol{\lambda} = ((3,1), \emptyset, (6,6,5,3), (19,0))$



Start from the frame sequence $\mathrm{fs}(\boldsymbol{\lambda})$

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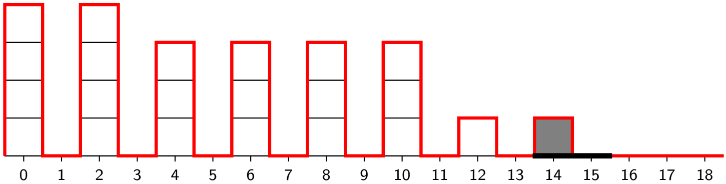

3 particle motions at index 14

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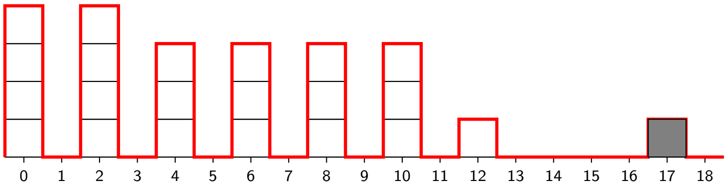

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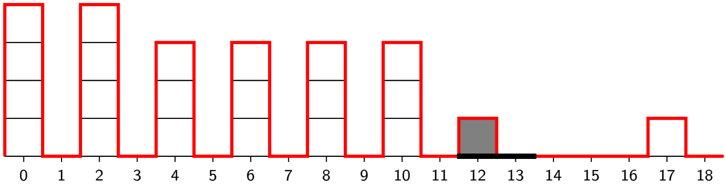

1 particle motion at index 12

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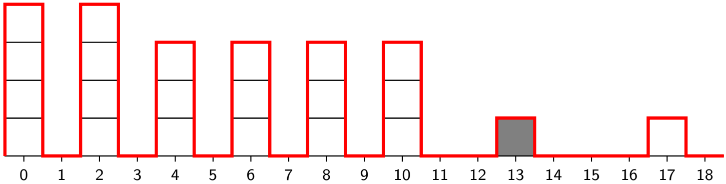

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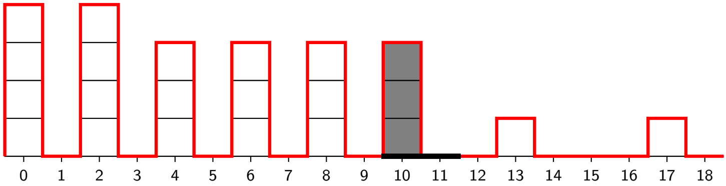

6 particle motions at index 10

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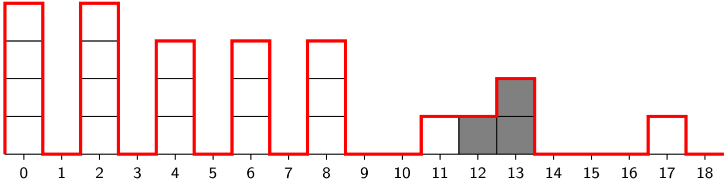

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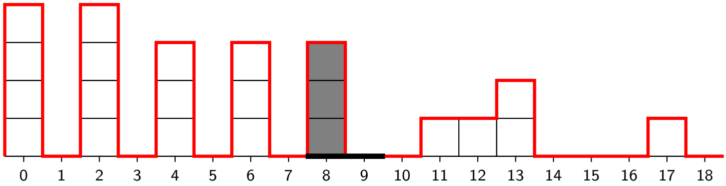

6 particle motions at index 8

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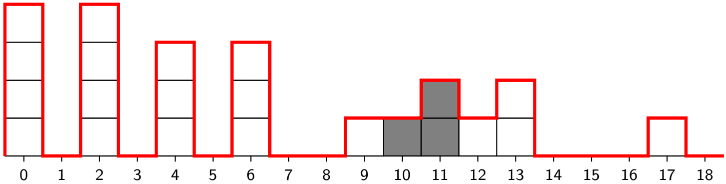

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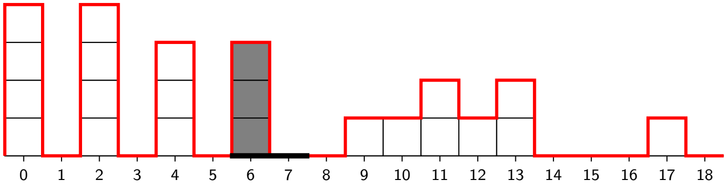

5 particle motions at index 6

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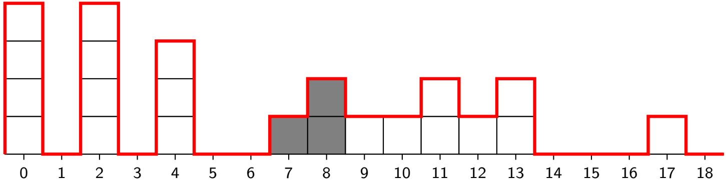

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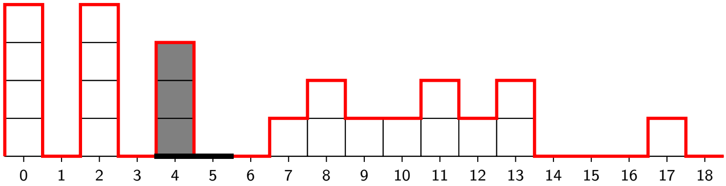

3 particle motions at index 4

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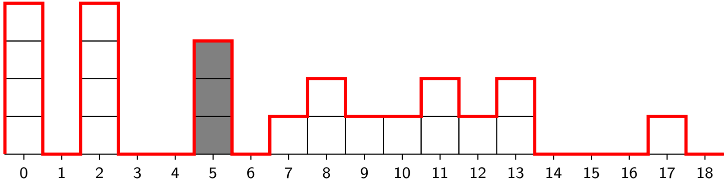

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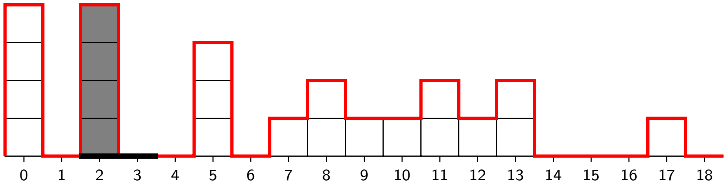

19 particle motions at index 2

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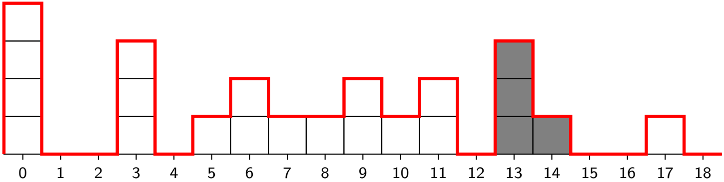

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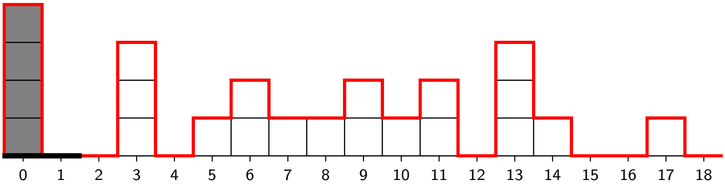

0 particle motions at index 0

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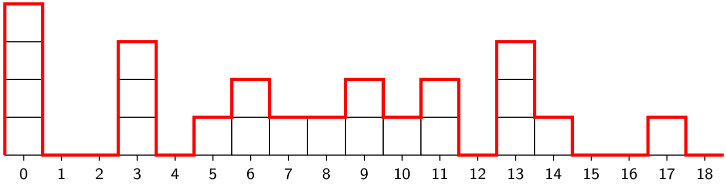

final result

Remarks

- If $\boldsymbol{\lambda}$ is a k-tuple of partitions, its frame sequence $\mathrm{fs}(\boldsymbol{\lambda})$ satisfies $f_i + f_{i+1} \leq k$ for all i, and the particle motion preserves that property. Thus $\Lambda(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})) \in \mathcal{A}_k$ for all $\boldsymbol{\lambda}$.

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- There is a procedure $\Gamma : \mathcal{A}_k \to \mathcal{P}_k$ which is the inverse of Λ. It is not hard to describe (each step is the inverse of the steps in the previous procedure), but proving that they are actually inverses of each other requires some calculations.

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- If we take

$$\boldsymbol{\lambda} := (\underbrace{(1,1,\ldots,1)}_{s_1-s_2}, \underbrace{(2,2,\ldots,2)}_{s_2-s_3}, \ldots, \underbrace{(k,k,\ldots,k)}_{s_k}),$$

then $\Lambda(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$ is Warnaar's minimal partition.

Consequence of the bijection

Because $\Lambda : \mathcal{P}_k \to \mathcal{A}_k$ is a weight-preserving bijection, we have

$$\sum_{(\boldsymbol{\lambda}, \mathsf{fs}(\boldsymbol{\lambda})) \in \mathcal{P}_k} q^{|\boldsymbol{\lambda}| + |\mathsf{fs}(\boldsymbol{\lambda})|} = \sum_{\lambda \in \mathcal{A}_k} q^{|\lambda|}.$$

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On one hand:

$$\begin{aligned}
\mathcal{A}_k &= \{(f_i)_{i \geq 0} : f_i + f_{i+1} \leq k \text{ for all } i \geq 0\} \\
&= \bigsqcup_{r=0}^{k} \{(f_i)_{i \geq 0} : f_i + f_{i+1} \leq k \text{ for all } i \geq 0 \text{ and } f_0 = r\} \\
&\sim \bigsqcup_{r=0}^{k} \{(f_i)_{i \geq 0} : f_i + f_{i+1} \leq k \text{ for all } i \geq 0 \text{ and } f_1 \leq k - r\} \\
&= \bigsqcup_{r=0}^{k} \mathcal{T}_{r,k}.
\end{aligned}$$

Consequence of the bijection

Hence, by the Andrews–Gordon identities,

$$\sum_{\lambda \in \mathcal{A}_k} q^{|\lambda|} = \sum_{r=0}^{k} \sum_{\lambda \in \mathcal{T}_{r,k}} q^{|\lambda|}$$

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Corollary (Bressoud 1980)

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2 + \cdots + s_k^2 - s_1 - \cdots - s_k}}{(q)_{s_1 - s_2} \ldots (q)_{s_{k-1} - s_k} (q)_{s_k}} = \sum_{r=0}^{k} \frac{(q^{2k+3}, q^{k-r+1}, q^{k+r+2}; q^{2k+3})_\infty}{(q)_\infty}.$$

Outline

1 Introduction

2 Warnaar's particle motion

3 Generalisation

4 Applications

General idea

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- Compute the generating function for the subset of $\mathcal{P}_k$ using the combinatorics of the pairs $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$. This gives an expression in the style of the sum-side of the Andrews–Gordon identities for the generating function of the subset of $\mathcal{A}_k$.

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- Compute the generating function for the subset of $\mathcal{P}_k$ using the combinatorics of the pairs $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$. This gives an expression in the style of the sum-side of the Andrews–Gordon identities for the generating function of the subset of $\mathcal{A}_k$.
- If one wants to prove Andrews–Gordon type identities, one can also find an expression for the generating function of the subset of $\mathcal{A}_k$ under consideration as a sum of products, using the Andrews–Gordon identities.

An important subset of $\mathcal{P}_k$

Definition

Let j, r, and k be non-negative integers such that $j + r \leq k$. Define

$$\mathcal{X}_{j,r,k} = \bigsqcup_{s_1 \geq \cdots \geq s_k \geq 0} \{\mathrm{fs}(s_1, \ldots, s_k)\} \times X_{j,r}(s_1, \ldots, s_k),$$

where $X_{j,r}(s_1, \ldots, s_k)$ is the set of k-tuples $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ s.t. $\forall m$,

- the length of $\lambda^{(m)}$ is $s_m - s_{m+1}$, and
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Proposition

$$\sum_{(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})) \in \mathcal{X}_{j,r,k}} q^{|(\mathrm{fs}(\boldsymbol{\lambda}), \boldsymbol{\lambda})|} = \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2 + \cdots + s_k^2 - (s_1 + \cdots + s_j) + (s_{k-r+1} + \cdots + s_k)}}{(q)_{s_1 - s_2} \cdots (q)_{s_{k-1} - s_k} (q)_{s_k}}.$$

Proof

Lemma

Let $P_{\ell,d}(n)$ be the number of partitions of n of length ℓ into parts at least d. Then

$$\sum_{n\geq 0} P_{\ell,d}(n)q^n = \frac{q^{d\ell}}{(q)_\ell}.$$

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Reminder: each part of $\lambda^{(m)}$ is at least $m - j + \max\{m-(k-r), 0\}$.

$$\sum_{\boldsymbol{\lambda}\in X_{j,r}(s_1,\dots,s_k)} q^{|\lambda|} = \prod_{m=1}^{j} \frac{1}{(q)_{s_m - s_{m+1}}} \times \prod_{m=j+1}^{k-r} \frac{q^{(m-j)(s_m - s_{m+1})}}{(q)_{s_m-s_{m+1}}}$$

$$\times \prod_{m=k-r+1}^{k-1} \frac{q^{(2m-k+r-j)(s_m-s_{m+1})}}{(q)_{s_m-s_{m+1}}} \times \frac{q^{(k+r-j)s_k}}{(q)_{s_k}}$$

$$= \frac{q^{(s_{j+1}+\cdots+s_{k-r})+2(s_{k-r+1}+\cdots+s_k)}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}}.$$

$\square$

Recovering the sum-side of Andrews–Gordon

Remark: $\mathcal{T}_{r,k} \subset \mathcal{A}_k$.

Theorem (D.–Jouhet–Konan 2024)

The map Λ induces a bijection between $\mathcal{X}_{0,r,k}$ and $\mathcal{T}_{r,k}$.

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$$\begin{cases} m & \text{if } m \leq k-r, \\ 2m-k+r & \text{if } m > k-r. \end{cases}$$

The minimal partitions of Warnaar correspond to the minimal partitions satisfying these conditions.

Stanton's identities

Theorem (Stanton 2018)

Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\dots(q)_{s_{k-1}-s_k}(q)_{s_k}}$$

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For $j = 0$ this reduces to Andrews–Gordon, while for $r = 0$ it yields the other identity of Bressoud proved via our bijection.

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Answer (D.–Jang–Jouhet 2025) : Yes!

Proof

- Recall that

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$$\mathcal{Z}_{j,r,k} = \{(f_i)_{i\geq 0} : f_i+f_{i+1} \leq k \; \forall i \geq 0,\; f_0 \leq j-\max\{f_0+f_1-(k-r),0\}\}.$$

  But this is not immediately related to Andrews–Gordon.
- We give a (not length-preserving, but weight-preserving) bijection between $\mathcal{Z}_{j,r,k}$ and some other set $\mathcal{Y}_{j,r,k}$ by modifying only the frequencies of 0.

Proof

- This new set $\mathcal{Y}_{j,r,k}$ is more clearly related to Andrews–Gordon. Indeed,

$$\mathcal{Y}_{j,r,k} = \{(f_i)_{i\geq 0} : f_i + f_{i+1} \leq k \ \forall i \geq 0 \\ \text{and } f_0 \in \{\ell + \max\{\ell - (j-r), 0\} : 0 \leq \ell \leq j\}\},$$

which can be written as a disjoint union of sets

$$Y_{s,k} = \{(f_i)_{i\geq 0} : f_i + f_{i+1} \leq k \ \forall i \geq 0 \text{ and } f_0 = s\}.$$

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- By Andrews–Gordon,

$$\sum_{f \in Y_{s,k}} q^{|f|} = \sum_{\lambda \in T_{s,k}} q^{|\lambda|} = \frac{(q^{2k+3}, q^{k+1-s}, q^{k+2+s}; q^{2k+3})_\infty}{(q)_\infty},$$

and we deduce

$$\sum_{f \in \mathcal{Y}_{j,r,k}} q^{|f|} = \sum_{s=0}^{j} \frac{(q^{2k+3}, q^{k+1-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+3})_\infty}{(q)_\infty}.$$

Proof

- By the bijections,

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-(s_1+\cdots+s_j)+(s_{k-r+1}+\cdots+s_k)}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}}$$

$$= \sum_{(\boldsymbol{\lambda},\mathrm{fs}(\boldsymbol{\lambda}))\in\mathcal{X}_{j,r,k}} q^{|(\mathrm{fs}(\boldsymbol{\lambda}),\boldsymbol{\lambda})|}$$

$$= \sum_{f\in\mathcal{Z}_{j,r,k}} q^{|f|}$$

$$= \sum_{f\in\mathcal{Y}_{j,r,k}} q^{|f|}$$

$$= \sum_{s=0}^{j} \frac{(q^{2k+3}, q^{k+1-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+3})_\infty}{(q)_\infty}.$$

□

Bressoud's identities

Theorem (Bressoud 1980)

Let r and k be integers with $k \geq 1$ and $0 \leq r \leq k$. Then

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\ldots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \frac{(q^{2k+2},q^{k+1-r},q^{k+1+r};q^{2k+2})_\infty}{(q)_\infty}.$$

Combinatorial version

Let $\mathcal{U}_{r,k}(n)$ be the set of partitions of n whose frequency sequence satisfies

- $f_i + f_{i+1} \leq k$ for all i,
- $f_1 \leq k - r$
- if $f_i + f_{i+1} = k$ for some i, then $if_i + (i+1)f_{i+1} \equiv k - r \pmod 2$.

Let $\mathcal{F}_{r,k}(n)$ be the set of partitions of n into parts **not** congruent to $0, \pm(k-r+1)$ modulo $2k+2$. Then for all n, $|\mathcal{U}_{r,k}(n)| = |\mathcal{F}_{r,k}(n)|$.

D.–Jouhet–Konan 2024: proof of the sum-side via the bijection.

Stanton's identities, Bressoud style

Theorem (Stanton 2018)

Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+2}, q^{k+1-r+j-2s}, q^{k+1+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty}.$$

For $j = 0$ this reduces to Bressoud's identities, while for $r = 0$ it yields another identity of Bressoud proved via the bijection by D.–Jouhet–Konan.

Stanton's identities, Bressoud style

Theorem (Stanton 2018)

Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2 + \cdots + s_k^2 - s_1 - \cdots - s_j + s_{k-r+1} + \cdots + s_k}}{(q)_{s_1 - s_2} \cdots (q)_{s_{k-1} - s_k} (q^2; q^2)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+2}, q^{k+1-r+j-2s}, q^{k+1+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty}.$$

For $j = 0$ this reduces to Bressoud's identities, while for $r = 0$ it yields another identity of Bressoud proved via the bijection by D.–Jouhet–Konan.

D.–Jang–Jouhet 2025

This identity can also be proved via the bijection (using Bressoud's identity as ingredient).

Kurşungöz' identities

Theorem (Kurşungöz 2016)

Let $\widetilde{\mathcal{U}}_{r,k}$ be the set of partitions whose frequency sequence satisfies

- $f_i + f_{i+1} \leq k$ *for all* i,
- $f_1 \leq k - r$
- *if* $f_i + f_{i+1} = k$ *for some* i, *then* $if_i + (i+1)f_{i+1} \equiv k - r + 1 \pmod 2$.

$$\sum_{\lambda \in \widetilde{\mathcal{U}}_{r,k}} q^{|\lambda|} = \frac{1}{(1+q)(q)_\infty}((q^{2k+2}, q^{k+r}, q^{k-r+2}; q^{2k+2})_\infty$$

$$+ q(q^{2k+2}, q^{k+2+r}, q^{k-r}; q^{2k+2})_\infty).$$

Theorem (D.–Jouhet–Konan 2024)

$$\sum_{\lambda \in \widetilde{\mathcal{U}}_{r,k}} q^{|\lambda|} = \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2 + \cdots + s_k^2 + s_{k-r+1} + \cdots + s_{k-1} + 2s_k}}{(q)_{s_1 - s_2} \ldots (q)_{s_{k-1} - s_k} (q^2; q^2)_{s_k}}$$

New Stanton-type generalisation of Kurșungöz' identities

Theorem (D.–Jang–Jouhet 2025)

Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_{k-1}+2s_k}}{(q)_{s_1-s_2}\ldots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$

$$= \frac{1}{1+q}\sum_{s=0}^{j}\left(\frac{(q^{2k+2},q^{k+2-r+j-2s},q^{k+r-j+2s};q^{2k+2})_\infty}{(q)_\infty} + q\frac{(q^{2k+2},q^{k-r+j-2s},q^{k+2+r-j+2s};q^{2k+2})_\infty}{(q)_\infty}\right).$$

For $j = 0$ this reduces to Kurșungöz' identities, while for $r = 0$ it yields another identity of D.–Jouhet–Konan related to Kurșungöz' identities obtained via the bijection.

Identities with parity restrictions

Theorem (Andrews 2010)

Let $\mathcal{W}_{k,a}$ denote the set of frequency sequences $(f_i)_{i\geq 1}$ such that

- $f_i + f_{i+1} \leq k$ *for all* i,
- $f_1 \leq a$,
- f_i *is even if* i *is even (i.e. even parts appear an even number of times).*

For $k \geq a \geq 0$ *and* $k \equiv a \pmod 2$,

$$\sum_{f \in \mathcal{W}_{k,a}} q^{|f|} = \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2+2s_{a+1}+2s_{a+3}+\cdots+2s_{k-1}}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$

$$= \frac{(-q;q^2)_\infty (q^{a+1}, q^{2k+3-a}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

There are similar theorems with other parity conditions by Andrews (2010), Kurşungöz (2010), Kim-Yee (2013).

Identities with parity restrictions

Theorem (D.–Jang 2026, Stanton-type generalisation)

For non-negative integers a, b, and k with $2a + 2b \leq k$, let $\mathcal{Z}^e_{a,b,k}$ be the set of all frequency sequences $(f_i)_{i \geq 0}$ such that

- $f_i + f_{i+1} \leq k$ *for all* $i \geq 0$,
- $f_0 \leq 2a$ *and* $2f_0 + f_1 \leq k - 2b + 2a$,
- f_i *is even if* i *is even (i.e. even parts appear an even number of times).*

$$\sum_{f \in \mathcal{Z}^e_{a,b,k}} q^{|f|} = \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2 + \cdots + s_k^2 - 2(s_2 + s_4 + \cdots + s_{2a}) + 2(s_{k-2b+1} + s_{k-2b+3} + \cdots + s_{k-1})}}{(q^2;q^2)_{s_1 - s_2} \cdots (q^2;q^2)_{s_{k-1} - s_k} (q^2;q^2)_{s_k}}$$

$$= \sum_{s=0}^{a} \frac{(-q;q^2)_\infty (q^{k+1+2a-2b-4s}, q^{k+3-2a+2b+4s}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

When $a = 0$, this reduces to Andrews' Theorem.

We also have similar generalisations of the results with other parity conditions. All the proofs use particle motion.

What's next?

- Interpretation as a different type of lattice paths and finite versions (D.–Jang–Jouhet–Nadeau, work in progress)
- Open problem:

Theorem (Stanton 2018)

Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k} \frac{q^{-s_1-\cdots-s_j}(1+q^{s_1+s_2})(1+q^{s_2+s_3})\cdots(1+q^{s_{j-1}+s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}}$$

$$= \sum_{s=0}^{j} \binom{j}{s} \frac{(q^{2k+3}, q^{k+1-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+3})_\infty}{(q)_\infty}.$$

Can this identity be proved via particle motion?

Thank you very much!