A reflection principle for nonintersecting paths and lozenge tilings with free boundaries

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A lattice path and nonintersecting lattice paths

Consider a Locally Finite and Acyclic Directed Graph.
Choose two vertices u_1 and v_1 of the graph.

Q: How many lattice paths are there from u_1 to v_1?

Let us denote the answer to this question by $\mathrm{GF}[\mathscr{P}(u_1, v_1)]$.

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Now, we consider $2m$ vertices $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_m)$ of the graph.

Q: How many families of m paths are there that join u_i and v_i for $i = 1, \ldots, m$ and are nonintersecting?

Let us denote the answer to the latter question by $\mathrm{GF}[\mathscr{P}_0(\mathbf{u}, \mathbf{v})]$.

0 intersection!

Lindström–Gessel–Viennot / Karlin–McGregor theorem

Let $\mathrm{M}(\mathbf{u}, \mathbf{v})$ be the $m \times m$ matrix defined by

$$\mathrm{M}(\mathbf{u}, \mathbf{v}) = \Big[\mathrm{GF}[\mathscr{P}(u_i, v_j)] \Big]_{1 \le i,j \le m}$$

and we call this matrix the *path matrix from* $\mathbf{u}$ *to* $\mathbf{v}$.

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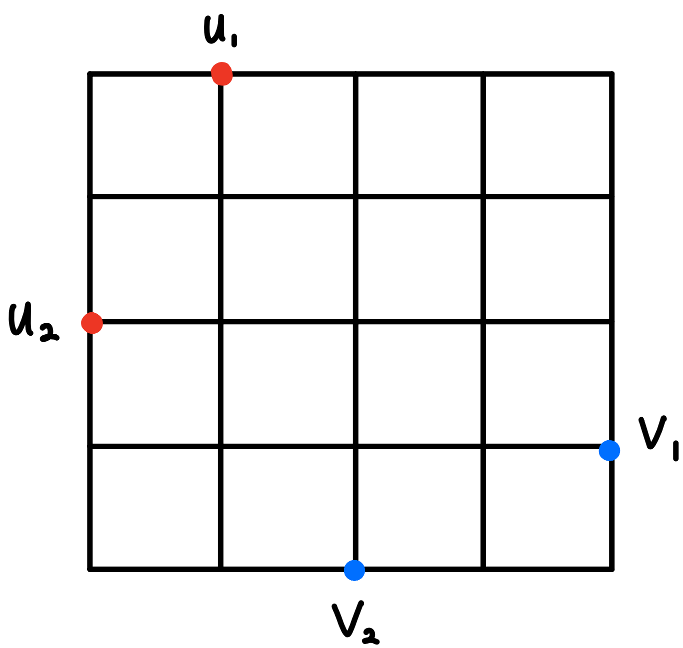
Theorem (Lindström–Gessel–Viennot, Karlin–McGregor)

If $\mathbf{u}$ and $\mathbf{v}$ are compatible, then

$$\mathrm{GF}[\mathscr{P}_0(\mathbf{u},\mathbf{v})] = \det \mathrm{M}(\mathbf{u},\mathbf{v}).$$

An example for Lindström–Gessel–Viennot theorem



$$\det M(u,v) = \begin{vmatrix} GF(u_1,v_1) & GF(u_1,v_2) \\ GF(u_2,v_1) & GF(u_2,v_2) \end{vmatrix}$$

$$= \begin{vmatrix} \binom{6}{3} & \binom{5}{4} \\ \binom{5}{1} & \binom{4}{2} \end{vmatrix}$$

$$= \begin{vmatrix} 20 & 5 \\ 5 & 6 \end{vmatrix}$$

$$= 120 - 25 = 95.$$

Families of paths with more ending points

Let $\mathbf{u} = (u_1, \ldots, u_m)$ be an m-tuple and $\mathbf{v} = (v_1, \ldots, v_n)$ be an n-tuple of points on G, where $n \geq m$. Assume that $\mathbf{u}$ and $\mathbf{v}$ are compatible.

Under these circumstances, we can still ask the same questions:

Q: How many families of m paths are there that join $u_1, \ldots, u_m$ and m vertices among $v_1 \ldots, v_n$ and are nonintersecting?

Okada–Stembridge theorem

Theorem (Okada–Stembridge)

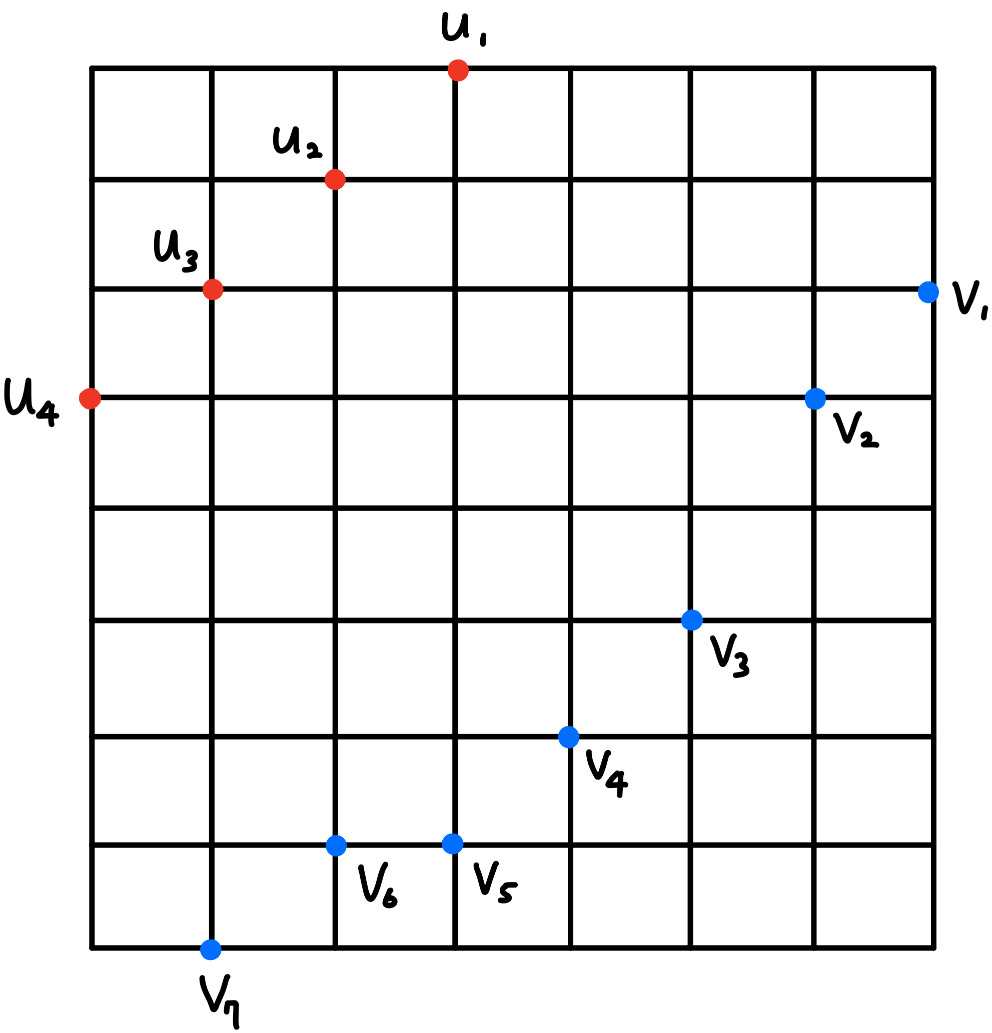
If m is even, then

$$\mathrm{GF}[\mathscr{P}_0(\mathbf{u},\mathbf{v})] = \mathrm{Pf}\left[\, \mathrm{Q}(\mathbf{u},\mathbf{v})\right],$$

where $\mathrm{Q}(\mathbf{u},\mathbf{v}) = \left[Q_{i,j}\right]_{1\le i,j\le m}$ is the skew-symmetric matrix defined by

$$Q_{i,j} = \sum_{s<t}\left[\,\mathrm{GF}[\mathscr{P}(u_i,v_s)]\cdot \mathrm{GF}[\mathscr{P}(u_j,v_t)] - \mathrm{GF}[\mathscr{P}(u_i,v_t)]\cdot \mathrm{GF}[\mathscr{P}(u_j,v_s)]\right].$$

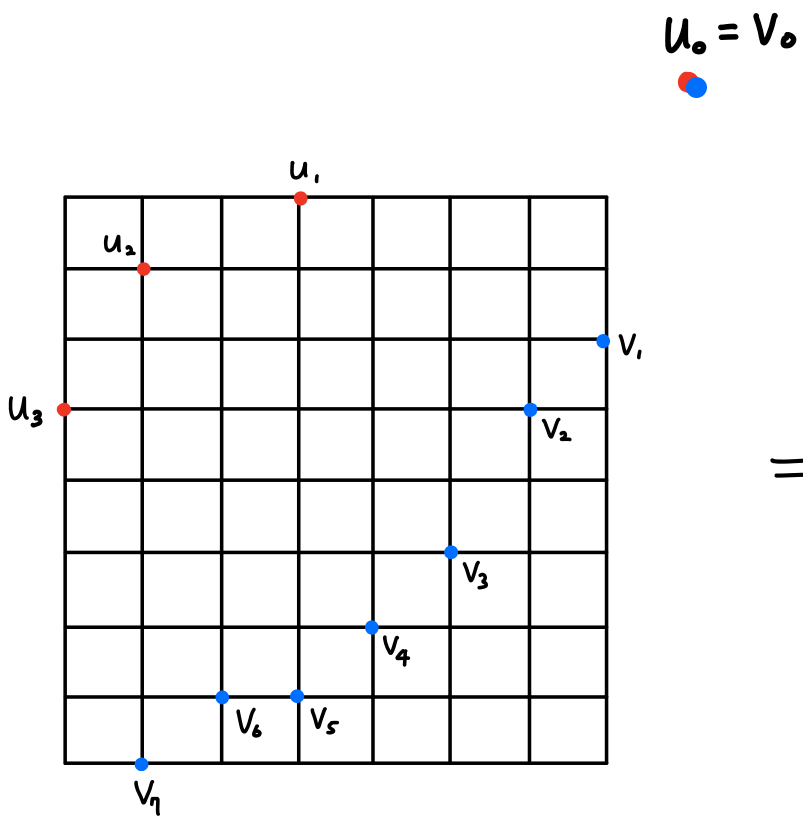
An example for Okada–Stembridge theorem



$$Q(u,v) = \begin{pmatrix} 0 & Q_{1,2} & Q_{1,3} & Q_{1,4} \\ -Q_{1,2} & 0 & Q_{2,3} & Q_{2,4} \\ -Q_{1,3} & -Q_{2,3} & 0 & Q_{3,4} \\ -Q_{1,4} & -Q_{2,4} & -Q_{3,4} & 0 \end{pmatrix}$$

$$Q_{1,2} = \left[\binom{6}{2}\cdot\binom{6}{2} - \binom{6}{3}\cdot\binom{6}{1}\right] + \left[\binom{6}{2}\cdot\binom{7}{4} - \binom{7}{5}\cdot\binom{6}{1}\right] + \cdots$$

Stembridge pointed out that this theorem can also be applied when m is odd by adding a "Phantom" vertex $u_0 = v_0$:



even!

Let $u' = (\overbrace{u_0, u_1, \cdots, u_m}^{m+1})$ and $v' = (v_0, v_1, \cdots, v_n)$.

$\Rightarrow$ Then $GF[\mathcal{P}_0(u,v)] = \underline{GF[\mathcal{P}_0(u',v')]}$

can apply Okada–Stembridge!

In this case, the resulting skew-symmetric matrix has order $m + 1$.

A natural question

While there is no parity condition on m (the number of starting points) in Lindström–Gessel–Viennot theorem, m should be even to apply the Okada-Stembridge theorem directly.

One can still apply Okada–Stembridge's theorem when m is odd, following Stembridge's remark, but the the resulting skew-symmetric matrix has a different order than when m is even.

Q: Is there an alternative formula that does not depend on the parity of m?

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Q: Is there an alternative formula that does not depend on the parity of m?

A: Yes.

A new theorem

Theorem (Byun, 2025)

We have

$$\mathrm{GF}[\mathscr{P}_0(\mathbf{u},\mathbf{v})]^2 = \det\left[\mathrm{M}(\mathbf{u},\mathbf{v})U_n\,\mathrm{M}(\mathbf{u},\mathbf{v})^T\right] = \det\left[\mathrm{M}(\mathbf{u},\mathbf{v})U_n^T\,\mathrm{M}(\mathbf{u},\mathbf{v})^T\right],$$

where $\underline{\mathrm{M}(\mathbf{u},\mathbf{v})}$ is the path matrix and $U_n = [u_{i,j}]_{1\le i,j\le n}$ is the upper

m×n matrix!

triangular matrix defined by $u_{i,j} = \begin{cases} 2, & \text{if } i<j \\ 1, & \text{if } i=j \\ 0, & \text{if } i>j \end{cases}$.

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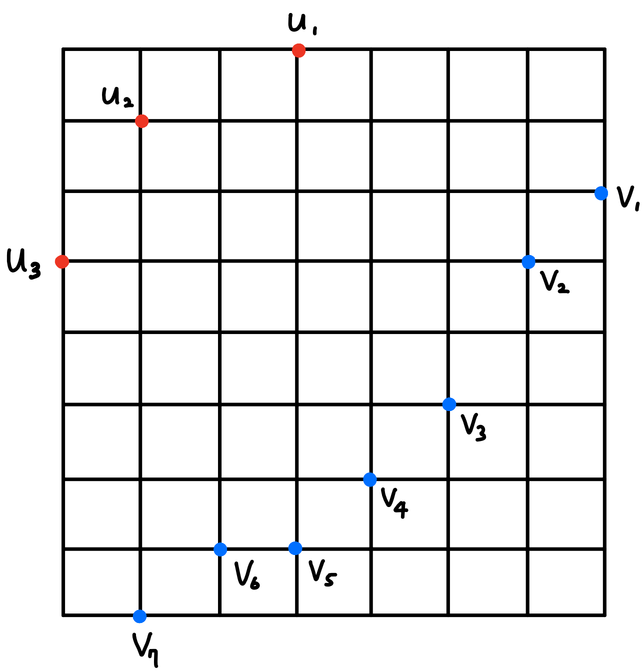
where $\mathrm{M}(\mathbf{u},\mathbf{v})$ is the path matrix and $U_n = [u_{i,j}]_{1\le i,j\le n}$ is the upper triangular matrix defined by $u_{i,j} = \begin{cases} 2, & \text{if } i<j \\ 1, & \text{if } i=j \\ 0, & \text{if } i>j \end{cases}.$

Note: there is no parity condition on m!

Main ingredients of Proof: (Lindström–Gessel–Viennot theorem) and (Okada's result on the sum of maximum minors of a matrix).

An example for the new theorem



$$M(u,v) = \begin{bmatrix} \binom{6}{2} & \binom{6}{3} & \binom{7}{5} & \binom{7}{6} & \binom{7}{7} & 0 & 0 \\ \binom{7}{1} & \binom{7}{2} & \binom{8}{4} & \binom{8}{5} & \binom{8}{6} & \binom{7}{6} & \binom{7}{7} \\ 0 & \binom{6}{0} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{6}{4} & \binom{6}{5} \end{bmatrix}$$

$$U_7 = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{M(u,v)\, U_7\, M(u,v)^T} = \begin{bmatrix} 4096 & 18769 & 13259 \\ 5551 & 36100 & 31100 \\ 1205 & 11840 & 12769 \end{bmatrix}$$

$$\det[\quad] = 47527052049 = 218007^2.$$

Hence, $GF[\mathcal{P}_0(u,v)] = 218007.$

A combinatorial interpretation of the new result

We can give a combinatorial interpretation of the new result, which we call "A reflection principle for nonintersecting paths."

This is obtained by interpreting $\mathrm{M}(\mathbf{u}, \mathbf{v})$, U_n, and $\mathrm{M}(\mathbf{u}, \mathbf{v})^T$ as path matrices of certain acyclic directed graphs.

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This is obtained by interpreting $\mathrm{M}(\mathbf{u}, \mathbf{v})$, U_n, and $\mathrm{M}(\mathbf{u}, \mathbf{v})^T$ as path matrices of certain acyclic directed graphs.

Consider a locally finite and acyclic directed graph G and an m-tuple $\mathbf{u} = (u_1, \ldots, u_m)$ and an n-tuple $\mathbf{v} = (v_1, \ldots, v_n)$ of vertices on G, where $m \leq n$.

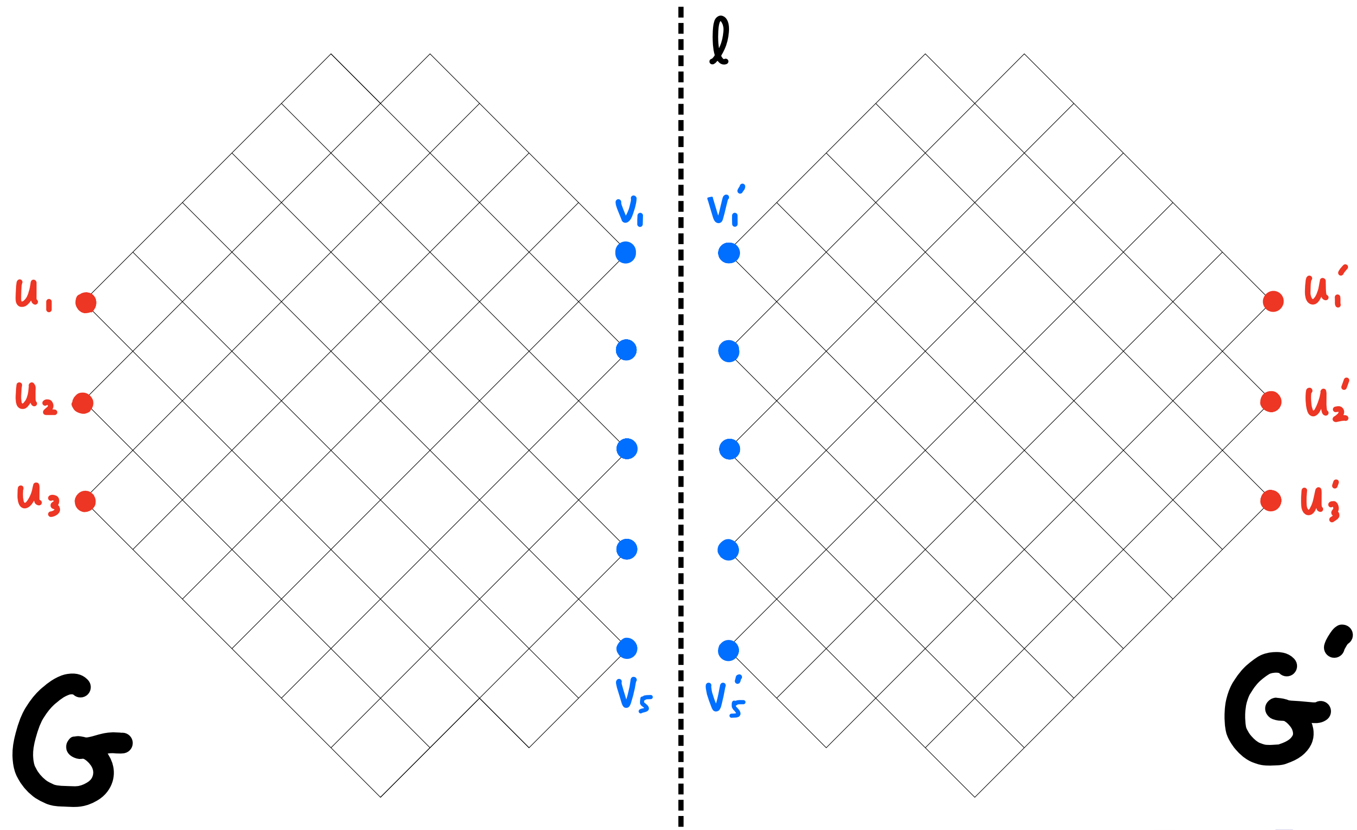
Assume further that the vertices $v_1, \ldots, v_n$ are sinks (that is, there are no outgoing edges from these n vertices in G).

A combinatorial interpretation of the new result (continue)

Reflect G and call the mirror image G'. We reverse the orientation of all edges of G'.

Denote the mirror images of $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ by $\mathbf{u}' = (u'_1, \ldots, u'_m)$ and $\mathbf{v}' = (v'_1, \ldots, v'_n)$, respectively, and $G_{sym} := G \cup G'$.

Note that $v'_1, \ldots, v'_n$ are sources of G'.



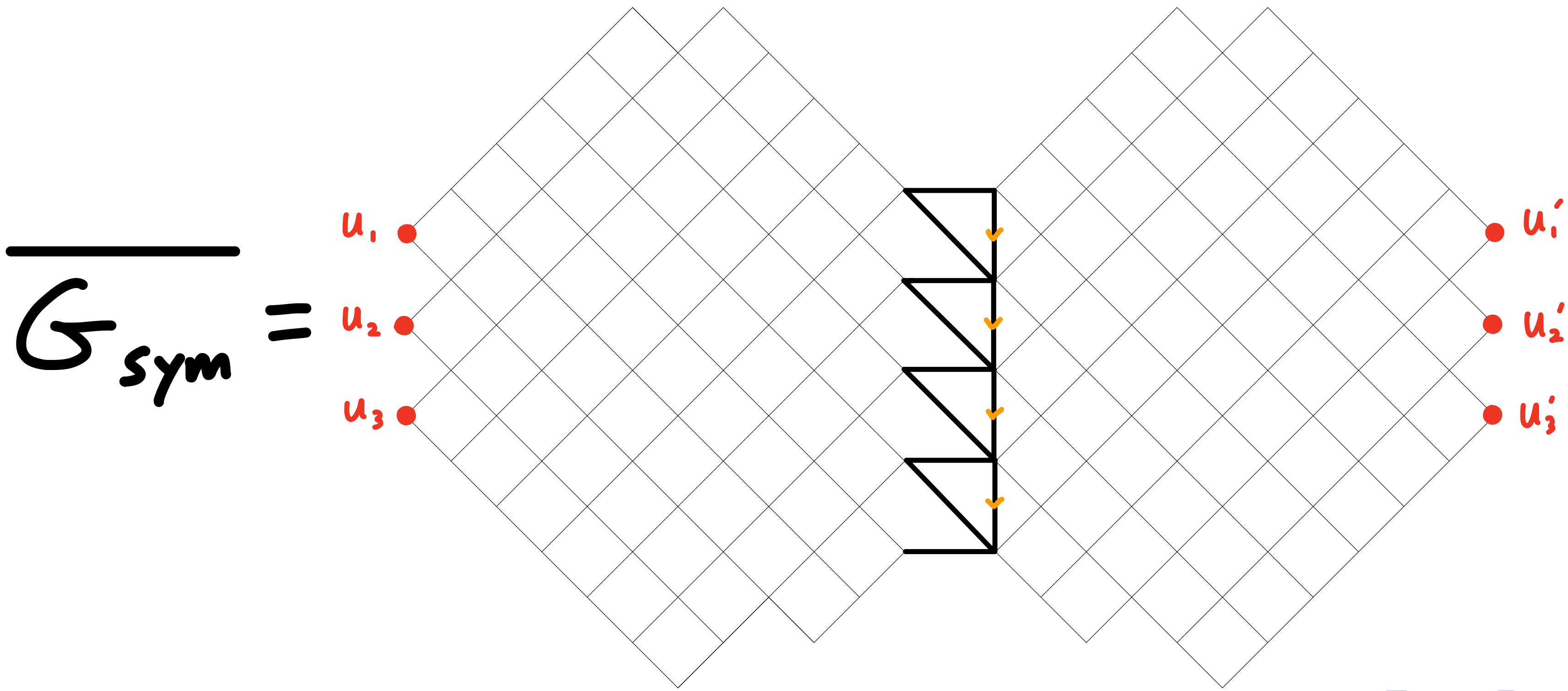
Construction of $\overline{G_{sym}}$

To construct $\overline{G_{sym}}$, from G_{sym}, we add

(1) n edges connecting v_i and v_i' for $i \in [n]$, directed toward v_i',

(2) $(n-1)$ edges connecting v_i and v_{i+1}' for $i \in [n-1]$, directed toward v_{i+1}', and

(3) $(n-1)$ edges connecting v_i' and v_{i+1}' for $i \in [n-1]$, directed toward v_{i+1}'



A reflection principle for nonintersecting paths

Let $\mathrm{GF}[\overline{\mathscr{P}_0}(\mathbf{u},\mathbf{u}')]$ be the number of m-tuples of nonintersecting paths $(\overline{P_1},\ldots,\overline{P_m})$ in $\overline{G_{sym}}$ such that $\overline{P_i}\in\mathscr{P}(u_i,u_i')$ for $i\in[m]$.

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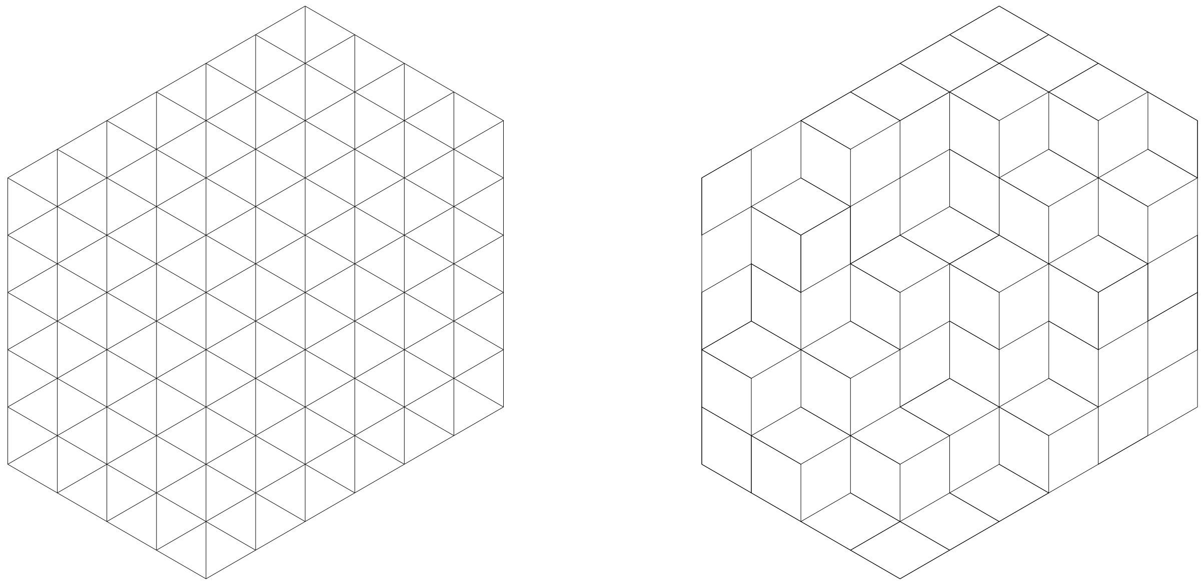
$$\mathrm{GF}[\mathscr{P}_0(\mathbf{u}, \mathbf{v})]^2 = \mathrm{GF}[\overline{\mathscr{P}_0}(\mathbf{u}, \mathbf{u}')].$$

$\Rightarrow$ The enumeration of families of nonintersecting paths <u>with unfixed ending points</u> can be resolved by enumerating families of nonintersecting paths <u>with fixed ending points</u> instead!

Lozenge tilings of regions on a triangular grid

Consider an arbitrary region R on a triangular grid, then one can ask the following question:



Q: How many lozenge tilings does R have?

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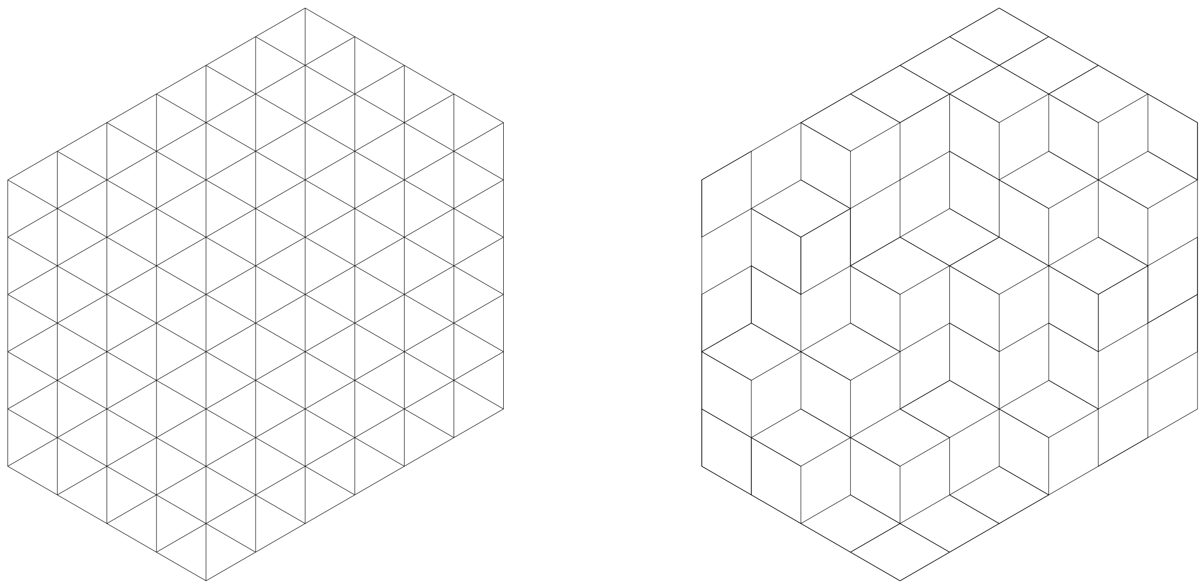


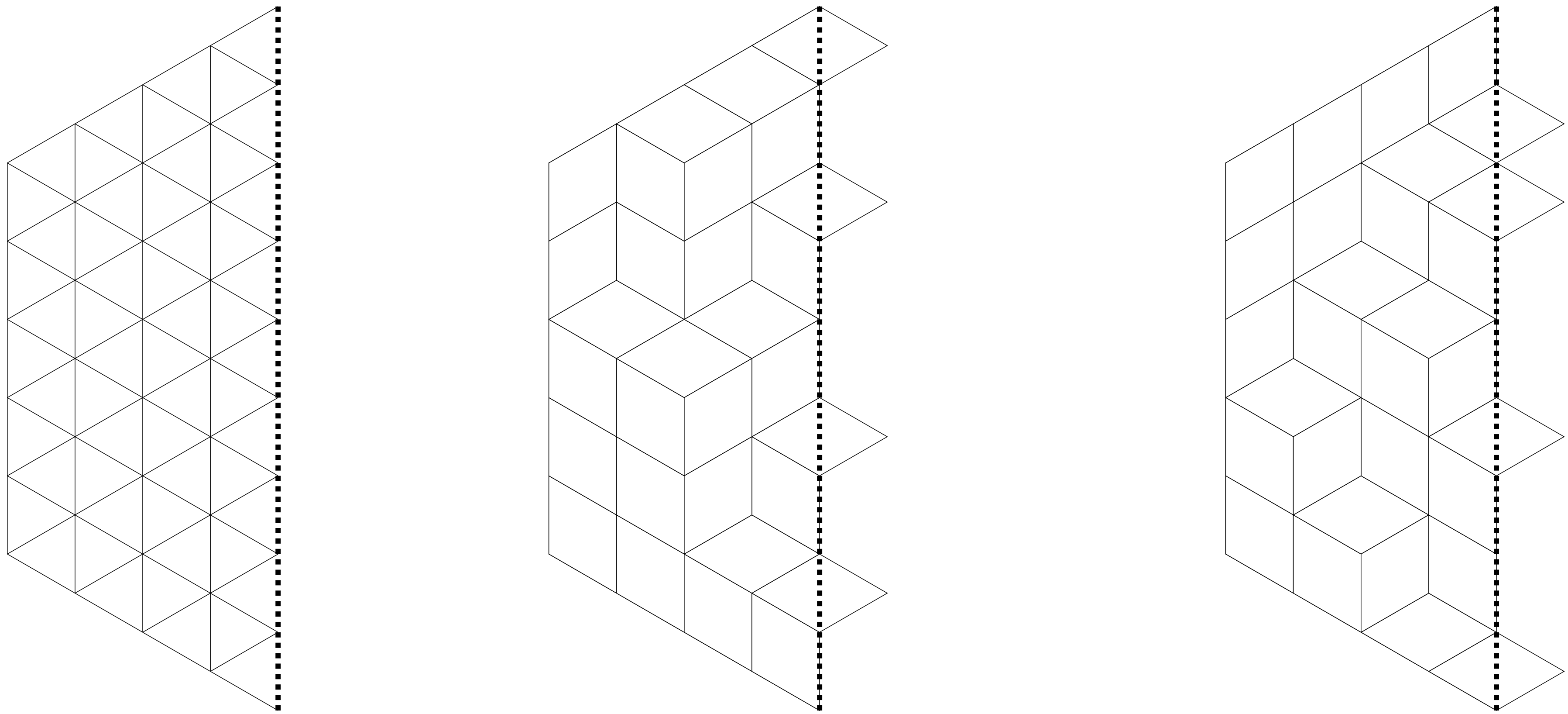
Q: How many lozenge tilings does R have?

It turns out that, in many cases, this question can be answered using nonintersecting paths enumeration techniques!

Lozenge tilings of regions with free boundaries

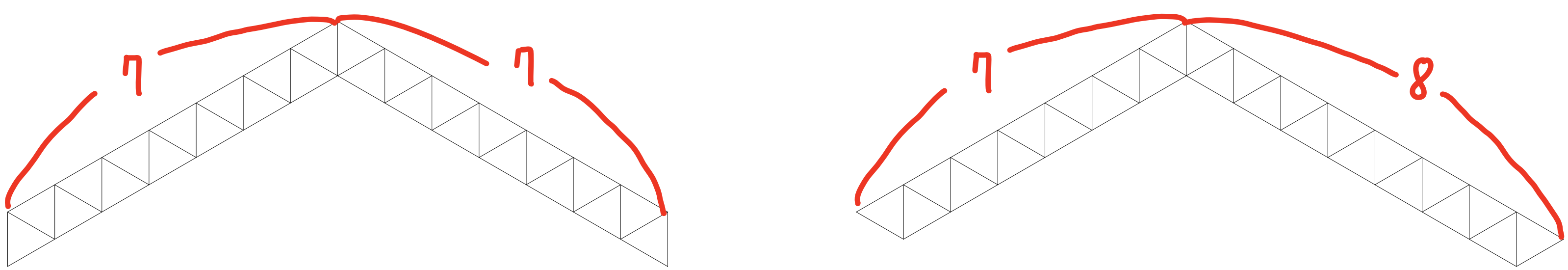
Lozenge tiling enumeration problems become more interesting if a region R has free boundaries!

A boundary is *free* if lozenges are allowed to cross the boundary.



A $\wedge$-hook and a shifted $\wedge$-hook

A *$\wedge$-hook of order n* is a $\wedge$-shaped hook that consists of $2n$ unit lozenges as described below.
A *shifted $\wedge$-hook of order n* is also a $\wedge$-shaped hook that is obtained from the $\wedge$-hook of order n by shifting the left-most unit triangle to the right end of it.



The $\wedge$-hook of order 7 (left) and the shifted $\wedge$-hook of order 7 (right).

Construction of two regions $A(m; \lambda_{st})$ and $\widetilde{A}(m; \lambda_{st})$

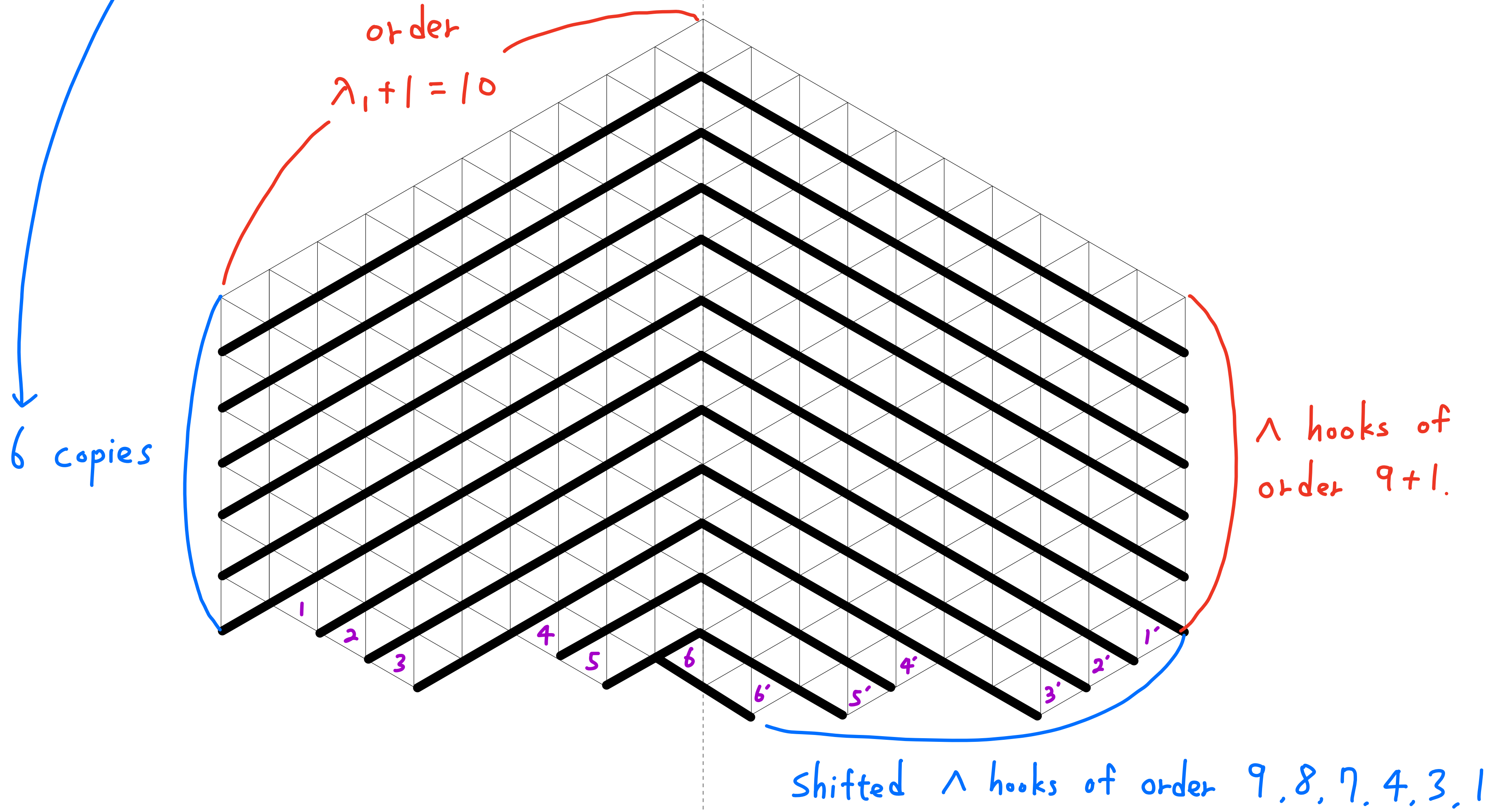
For any $m \in \mathbb{Z}_{\geq 0}$ and a strict partition $\lambda_{st} = (\lambda_1, \ldots, \lambda_k)$, consider m copies of $\wedge$-hook of order $(\lambda_1 + 1)$ and a copy of shifted $\wedge$-hook of order λ_i for $i \in [k]$.

(m copies) (k copies)

Then we concatenate these $\wedge$-hooks and shifted $\wedge$-hooks in order from top to bottom along the common axis ℓ, starting with the largest one in weakly decreasing order.

Example: $m = 6$ and $\lambda_{st} = (9, 8, 7, 4, 3, 1)$.

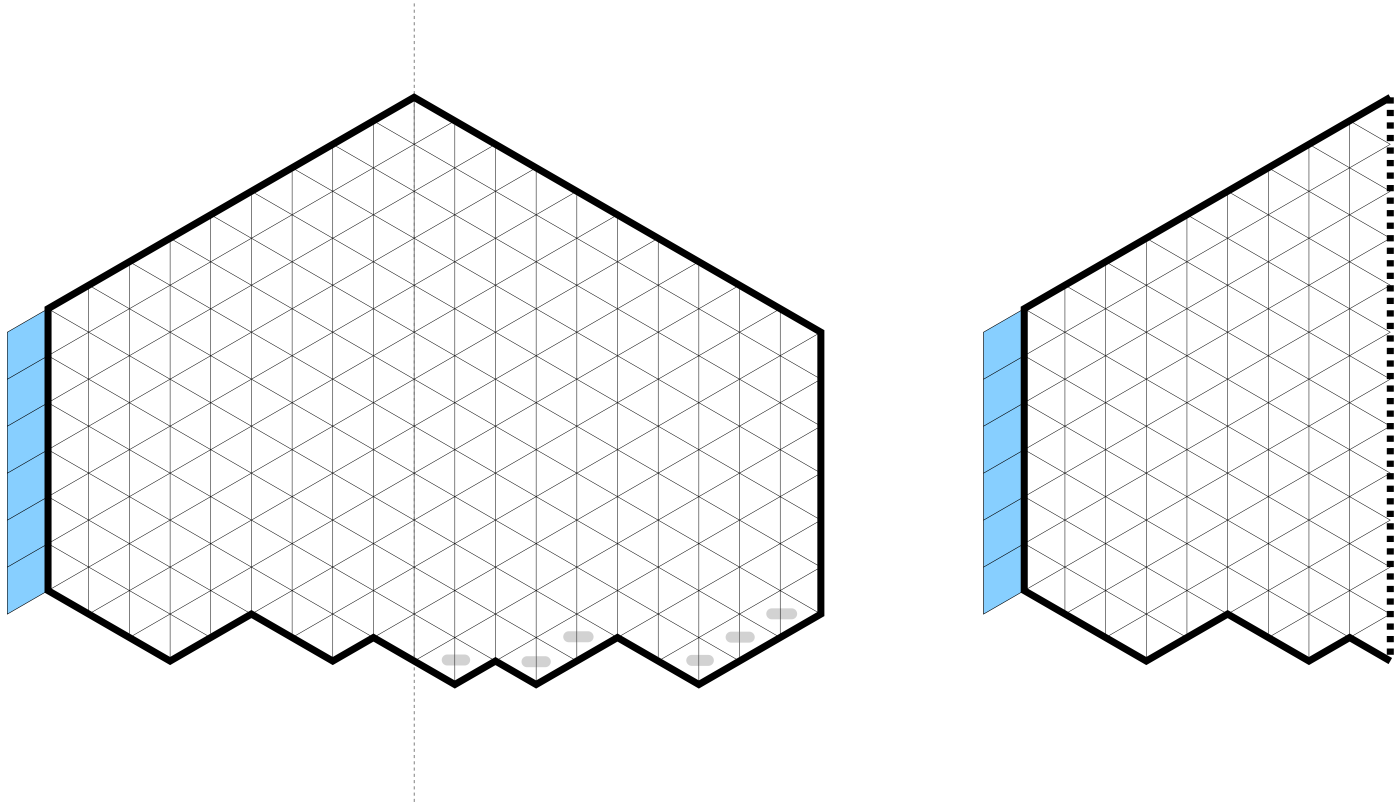
λ_1



Two regions $\widetilde{A}(m;\lambda_{st})$ and $A(m;\lambda_{st})$

If we get rid of the leftmost strip from the previous regions, it is $\widetilde{A}(m;\lambda_{st})$. Every lozenge is weighted by 1, except the k horizontal lozenges at the right end of k shifted $\wedge$-hook of order λ_i for $i \in [k]$.

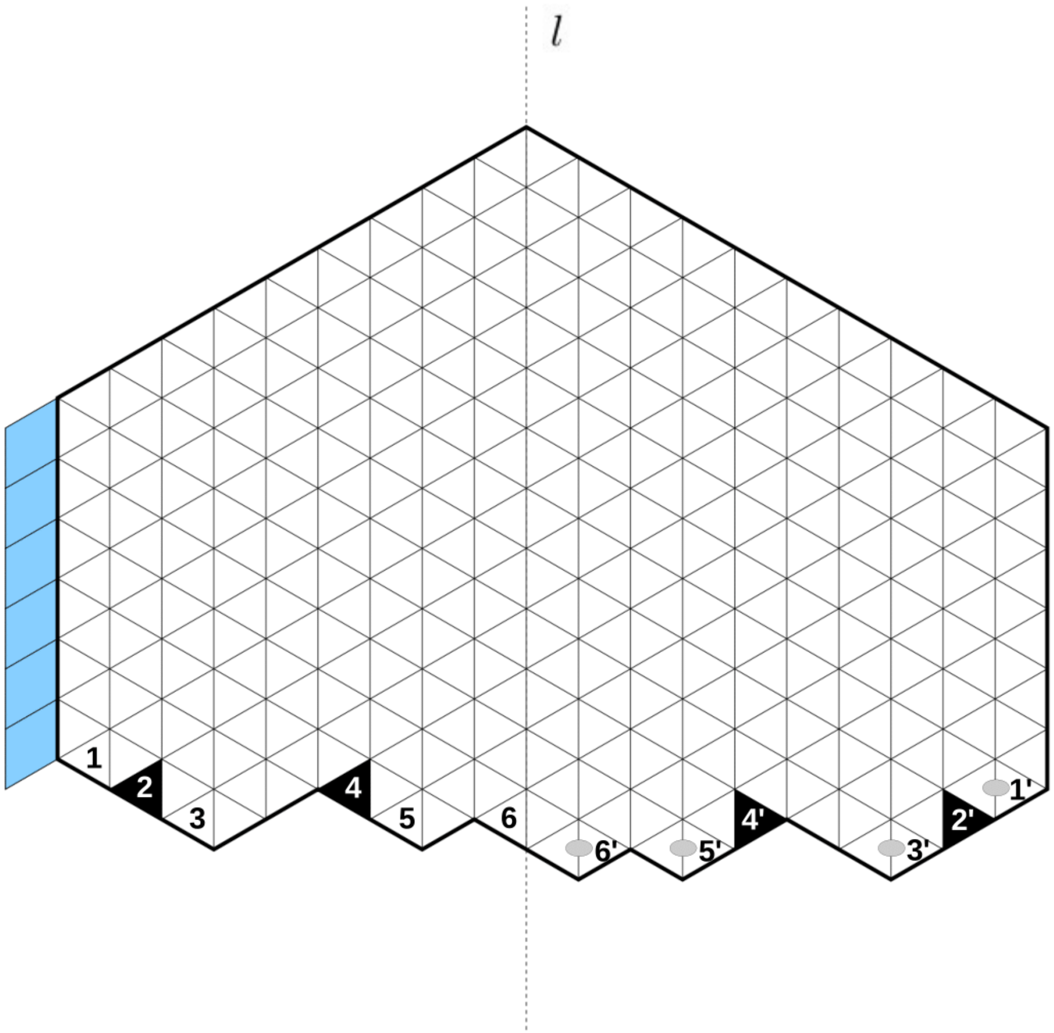
they are weighted by $\frac{1}{2}$.

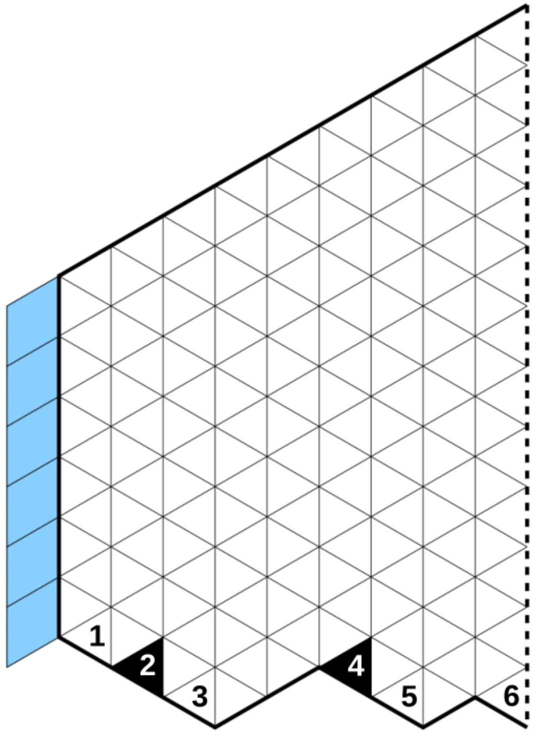


The subregion left to ℓ is $A(m;\lambda_{st})$ and its boundary along ℓ is free.

More general regions $\widetilde{A}(m; \lambda_{st}; I)$ and $A(m; \lambda_{st}; I)$

$m = 6$, $\lambda_{st} = (9, 8, 7, 4, 3, 1)$, and $I = \{2, 4\}$.





A new theorem

Let $\mathrm{M}_f(A(m; \lambda_{st}; I))$ be the number of lozenge tilings of $A(m; \lambda_{st}; I)$.

Let $\mathrm{M}(\widetilde{A}(m; \lambda_{st}; I))$ be the tiling generating function of $\widetilde{A}(m; \lambda_{st}; I)$.

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Let $\mathrm{M}(\widetilde{A}(m;\lambda_{st};I))$ be the tiling generating function of $\widetilde{A}(m;\lambda_{st};I)$.

Theorem (Byun, 2025)

For a nonnegative integer m, a strict partition λ_{st} with k parts, and a set $I \subseteq [k]$,

$$\mathrm{M}_f(A(m;\lambda_{st};I))^2 = 2^{k-|I|}\,\mathrm{M}(\widetilde{A}(m;\lambda_{st};I)).$$

Proof idea: The proof uses 1) a bijection between lozenge tilings and nonintersecting paths and 2) our earlier determinant formula.

Thank you!