A differential analogue of Tutte's invariant method

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Joint work with Charlotte Hardouin

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The enumeration of lattice walks
restricted to (non-)convex cones

WALKS WITH SMALL STEPS IN THE QUARTER PLANE

MIREILLE BOUSQUET-MÉLOU AND MARNI MISHNA

ABSTRACT. Let $\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$. We address the enumeration of plane lattice walks with steps in $\mathcal{S}$, that start from $(0, 0)$ and always remain in the first quadrant $\{(i, j) : i \geq 0, j \geq 0\}$. *A priori*, there are 2^8 problems of this type, but some are trivial. Some others are equivalent to a model of walks confined to a half-plane: such models can be solved systematically using the kernel method, which leads to algebraic generating functions. We focus on the remaining cases, and show that there are 79 inherently different problems to study.

To each of them, we associate a group G of birational transformations. We show that this group is finite (of order at most 8) in 23 cases, and infinite in the 56 other cases.

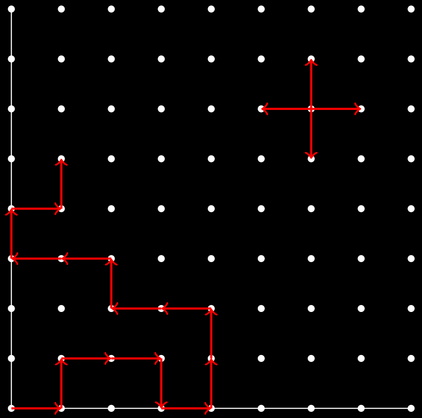
We present a unified way of solving 22 of the 23 models associated with a finite group. For all of them, the generating function is found to be D-finite. The 23rd model, known as Gessel's walks, has recently been proved by Bostan *et al.* to have an algebraic (and hence D-finite) solution. We conjecture that the remaining 56 models, associated with an infinite group, have a non-D-finite generating function.

Our approach allows us to recover and refine some known results, and also to obtain new results. For instance, we prove that walks with N, E, W, SS, SW and NE steps have an algebraic generating function.

1. Introduction

The enumeration of lattice walks is a classical topic in combinatorics. Many combinatorial objects (trees, maps, permutations, lattice polygons, Young tableaux, queues...) can be encoded by lattice walks, so that lattice path enumeration has many applications. Given a lattice, for instance the hypercubic lattice $\mathbb{Z}^d$, and a finite set of steps $\mathcal{S} \subset \mathbb{Z}^d$, a typical problem is to determine how many n-step walks with steps taken from $\mathcal{S}$, starting from the origin, are confined to a certain region $\mathcal{A}$ of the space. If $\mathcal{A}$ is the whole space, then the length generating function of these walks is a simple rational series. If $\mathcal{A}$ is a half-space, bounded by a rational hyperplane, the associated generating function is an algebraic series. Instances of the latter

Ayyer, Bacher, Banderier, Beaton, Bernardi, Bogosel, Bonnet, Bostan, Bousquet-Mélou, Buchacher, Chapon, Chyzak, Courtiel, D'Arco, Denisov, Di Vizio, Dreyfus, Drmota, Elvey Petkovšek, Prellberg, Price, Ponty, Fayolle, Flajolet, Fusy, Gohier, Hainzl, Hardouin, Hauser, Hillairet, Hoang, Humbert, Iasnogorodski, Johnson, Jímenez-Pastor, Jenne, Kauers, Koutschan, Kuba, Kurkova, Malyshev, Lacivita, Lairez, Lamali, Lumbroso, Marcovici, Melczer, Mishna, Mustapha, Nessmann, Notarantonio, Owczarek, Panafieu, Perrollaz, Pulido, Raschel, Rechnitzer, Roques, Salvy, Singer, Trotignon, Van Hoeij, Wachtel, Wallner, Wilson, Wong, Xu, Yatchak, Yeats, Zeilberger



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are neither trivial nor hopelessly difficult to solve.

They can be approached by
analytic and algebraic methods.

We discuss Tutte's invariant method [1], explain what it relies on, and introduce a differential analogue of it.

[1] *Counting quadrant walks via Tutte's invariant method*, Bernardi, Bousquet-Mélou, Raschel

A lattice walk is a sequence $P_0, P_1, \dots, P_n$ of points of $\mathbb{Z}^d$.
We call P_0 and P_n its starting and end point, respectively, the consecutive differences $P_{i+1} - P_i$ are its steps, and n is its length.

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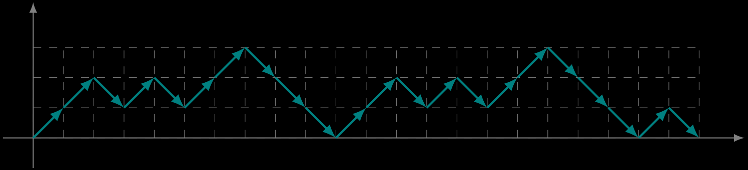

Let $f(i;n)$ be the number of walks on $\mathbb{N}$ that start at 0, end at i and consist of n steps all of which are taken from $\{-1, 1\}$, and let

$$F(x;t) := \sum_{n\geq 0} \left(\sum_{i \geq 0} f(i;n) x^i \right) t^n$$

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be the associated generating function. Then $F(x;t)$ is the unique solution of

$$x\left(1-t\left(x+x^{-1}\right)\right)F(x;t) = x - tF(0;t)$$

in $\mathbb{Q}[x][[t]]$.

To compute $f(i;n)$ and its asymptotics we locate $F(x;t)$ in the hierarchy of algebraic, D-finite and D-algebraic functions.

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The method relies on the invariant lemma and the construction of rational functions that are referred to as invariants.

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We claim that it allows us to derive from

$$K(x,t)F(x;t) = x - tF(0;t)$$

a non-trivial algebraic equation for $F(0;t)$.

Indeed, using that

$$-\frac{K(x,t)}{tx} = x + \frac{1}{x} - \frac{1}{t}$$

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and find that

$$1 - F(0;t) + t^2 F(0;t)^2$$

is a multiple of $K(x,t)$ in $\mathbb{Q}[[x,t]]$ and hence equal to zero.

A pair $(f, g) \in \mathbb{Q}(x) \times \mathbb{Q}(t)$ such that

$$q(x, t)K(x, t) = f(x) - g(t)$$

for some non-trivial $q \in \mathbb{Q}(x, t)$ is said to be an invariant.

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[4] — " — : *the local case*, Buchacher

[5] *Galoisian structure of large steps walks in the quadrant*, Bonnet, Hardouin

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There are two semi-algorithms [3, 4, 5] they can be computed with.

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Construction [6,7] of invariants

[6] *Separating variables in bivariate polynomial ideals*, Buchacher, Kauers, Pogudin
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Construction [6,7] of invariants

The non-linear problem of solving

$$qK = f - g$$

is reduced to a linear problem. The reduction is based on the computation of the poles of f and g and their multiplicities.

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Multiplicities

If ∞ is a pole of f and g, then there is an $\omega \in \mathbb{N}^2$ such that

$$\mathrm{lp}_\omega(q)\mathrm{lp}_\omega(K) = \mathrm{lp}_\omega(f) - \mathrm{lp}_\omega(g).$$

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It allows us to derive a differential equation
for $F(x; t)$ without relying on its algebraicity.

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is D-finite with respect to x we differentiate the equation to find

$$(-t + x - tx^2)\frac{\partial^2 F(x;t)}{\partial x^2} + 2(1 - 2tx)\frac{\partial F(x;t)}{\partial x} - 2tF(x;t) = 0.$$

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$$L = \left(\frac{1}{t^2} - 4\right)\partial^2 + \frac{1}{t}\partial - 1 \quad \text{with} \quad \partial = \frac{x^2}{x^2-1}\frac{\partial}{\partial x} - t^2\frac{\partial}{\partial t}.$$

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Since

$$L \cdot \langle K \rangle \subseteq \langle K \rangle, \quad L \cdot x \in \langle K \rangle \quad \text{and} \quad L \cdot t \in \mathbb{Q}(t),$$

it follows that

$$L \cdot (tF(0;t))$$

is a multiple of $K(x,t)$ in $\mathbb{Q}[[t]]$ and hence equal to zero.

Some differential algebra

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A derivation ∂ on a field E is a map $\partial : E \to E$ such that

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If $a \in F$ is algebraic over E, then ∂ extends uniquely to a. Furthermore, there is an $L \in E\langle\partial\rangle \setminus \{0\}$ such that

$$L \cdot a = 0.$$

Construction of $\mathbb{L}$

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If $\mathbb{Q}(x, t)$ denotes the function field of $K(x, t)$, then

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The derivation on $\mathbb{Q}(x,t)$ that maps $1/t$ to 1 is induced by

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Furthermore, x is algebraic over $\mathbb{Q}(x) \cap \mathbb{Q}(t)$, and hence satisfies a linear differential equation in ∂. It is described by L.

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It provides proofs of D-finiteness of walks whose step set is contained in $\{-1, 0, 1, 2\}^2$ not known to be D-finite before.

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Summary and open problem

Let $S(x, y)$ be a Laurent polynomial such that

$$\operatorname{supp}(S) \subseteq \{-1, 0, 1\}^2$$

and define

$$K(x, y) = xy(1 - tS(x, y)).$$

The nature of the solution of

$$\begin{aligned} K(x,y)F(x,y) &= xy + K(x,0)F(x,0) \\ &\quad + K(0,y)F(0,y) - K(0,0)F(0,0) \end{aligned}$$

in $\mathbb{Q}[x,y][[t]]$ depends on S and can be very diverse.

The following theorem is due to Bernardi, Bousquet-Mélou, Dreyfus, Elvey Price, Hardouin, Raschel, Singer.

Thm

It is D-finite iff

$$q(x, y)K = f(x) - g(y)$$

has a (non-trivial) solution in $\mathbb{Q}(t)(x, y)$. It is algebraic iff it is D-finite and

$$xy + q(x, y)K = f(x) - g(y)$$

has a solution. Furthermore, if it is not D-finite, then it is D-algebraic iff the latter equation has a solution.

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Thank you!