Progressive and rushed Dyck paths

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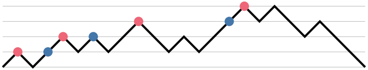
Definition

A Dyck path is progressive if every visit at height $h \geq 2$ is preceded by at least two visits at height $h - 1$.



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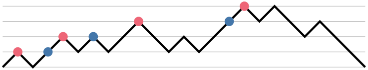

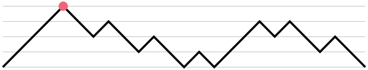

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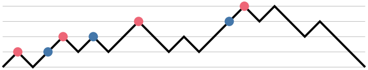
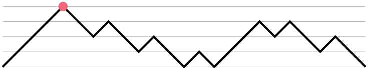


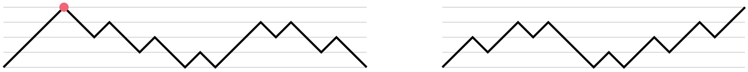


Theorem [Asinowski and Jelínek, A287709]

There are as many progressive as rushed paths of length $2n$.

Rushed paths and culminating paths

- A rushed path of height h and length $2n$ is equivalent to a culminating path ending at $(2n-h,h)$.
  [Bousquet-Mélou and Ponty, 2008]



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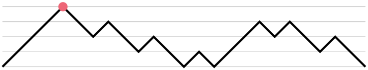

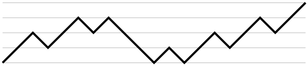

- The generating function of rushed paths is:

$$R(z) = \sum_{h=1}^{\infty} \frac{z^h}{F_h(z)}$$

where $F_h(z)$ is the Fibonacci polynomial.

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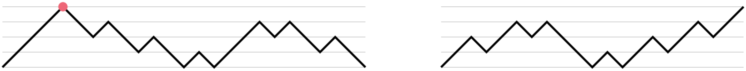

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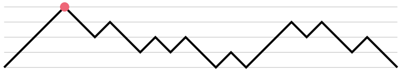
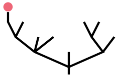
- Rushed and culminating paths of length n are different.

One-sided trees and asymptotics





Theorem **[Durhuus and Ünel, 2023]**

The number of one-sided trees (=rushed paths) satisfies:

$$r_n \sim \lambda 4^n e^{-\nu n^{1/3}} n^{-5/6}.$$

Their height satisfies:

$$\frac{h(R_n) - \mu n^{1/3}}{\sigma n^{1/6}} \xrightarrow{d} \mathcal{N}(0,1).$$

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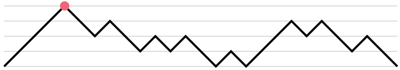

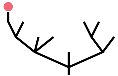

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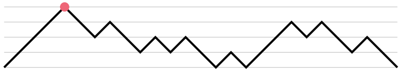

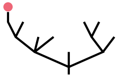

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- Dyck paths with weight 2^{-h} have the same asymptotics. **[Beaton and McKay, 2014; Durhuus and Ünel, 2023]**

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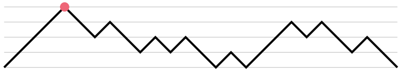

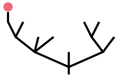

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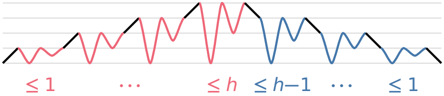
- Durhuus and Ünel give the local limit of one-sided trees.

Dyck paths ⟷ progressive culminating paths

- Start from a Dyck path of length $2n$ and height h and decompose it into downward factors.



Dyck paths ⟷ progressive culminating paths

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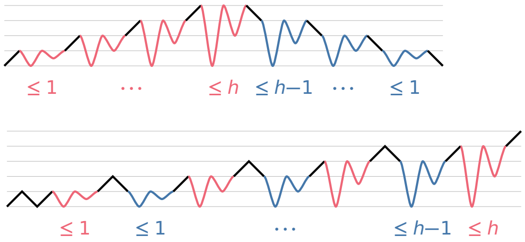
- Rearrange the factors into a progressive culminating path ending at $(2n+h+1, h+1)$.



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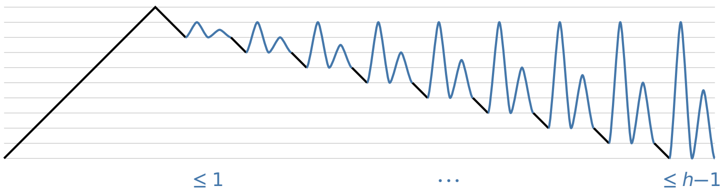

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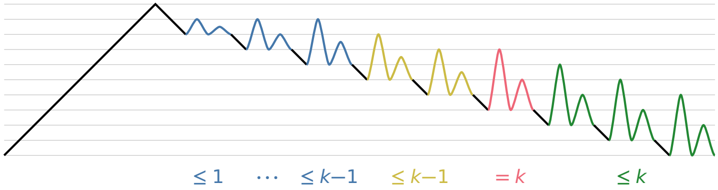
- Decompose the rushed path into upward factors.
- Find the leftmost highest factor



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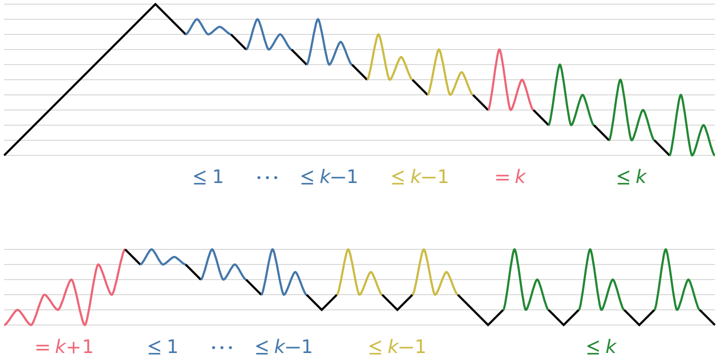

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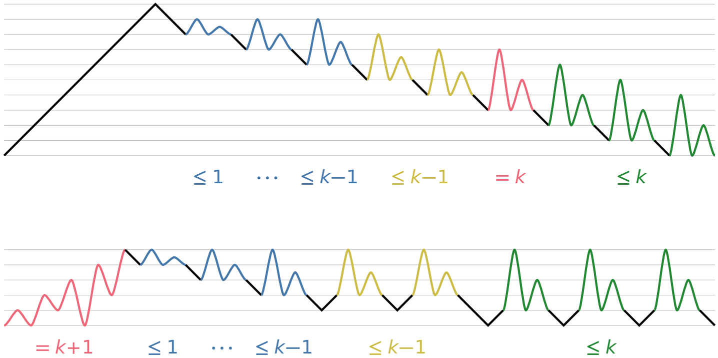
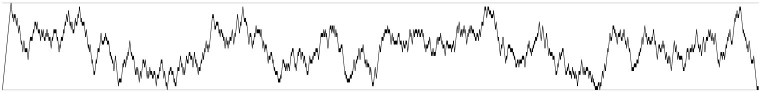


- The resulting progressive path has equal or lower height.

Arches and height of paths



- The number of arches of rushed paths tends to a discrete law, given by Durhuus and Ünel's local limit:

$$\mathbf{E}\left[z^{a(R)}\right] \to \frac{z}{2-z} 2^{\frac{z}{2-z}-1}.$$

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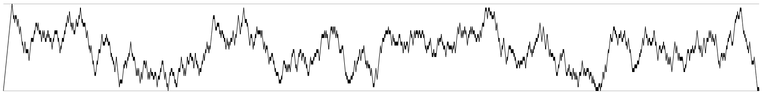

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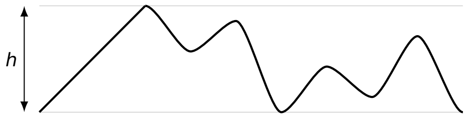
- The height difference of a rushed/progressive pair tends to a related discrete law:

$$\mathbf{E}\left[z^{h(R)-h(P)}\right] \to \frac{1}{z}\left(1 - \frac{1-z}{2-z}\, 2^{\frac{z}{2-z}+1}\right).$$

Random sampling of rushed paths



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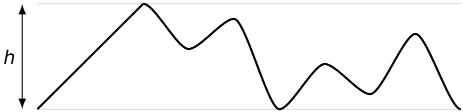
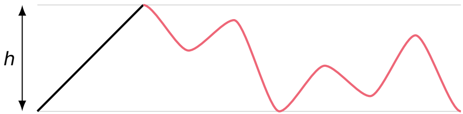


- Efficient random sampling of culminating paths is done in **[Bousquet-Mélou and Ponty, 2008]** but is not applicable.
- Instead, the strategy is:
  - select a height h with the right distribution,
  - sample a rushed path of height h.

Selecting the height



- We use ball arithmetic to compute $r_{n,h}$ on demand, with precision refined on demand.
  [Denise Zimmermann 1999; Bodini Durand Marchal 2024]

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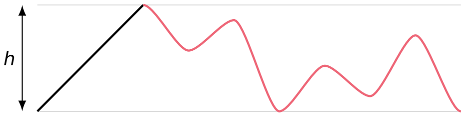

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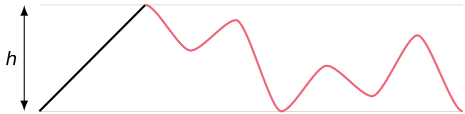

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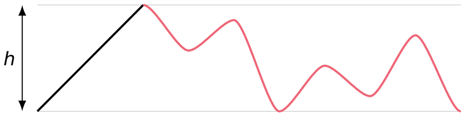

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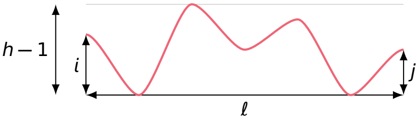
- With high probability, we only compute $r_{n,h}$ for $h \leq O(n^{1/3})$ to $O(\log n)$ bits of precision.
- Average complexity is $O(n \log^3 n)$. **[Bostan and Mori, 2021]**

Sampling a path of height h



- Fact: if $j \geq i$ and $j + i \leq h$, then:

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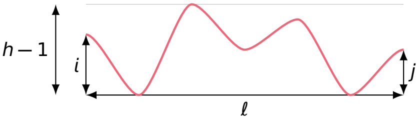

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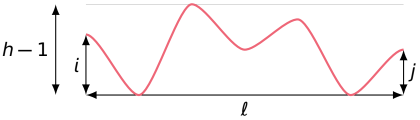

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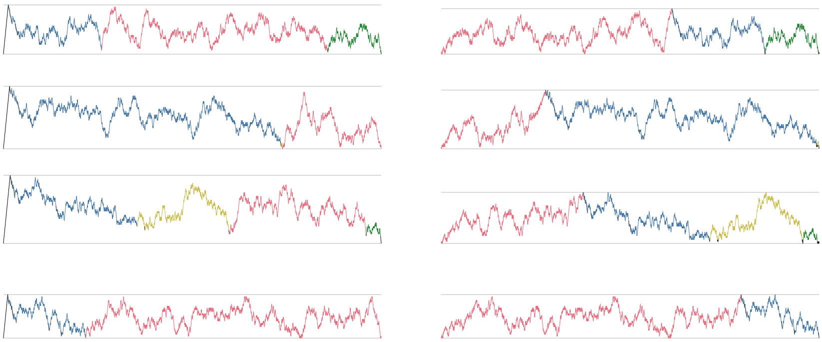
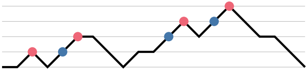
Random rushed path examples



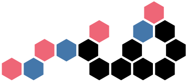
Perspectives

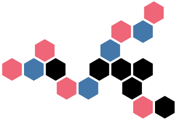
- Paths with different steps?



- Doubly progressive paths?

- Directed animals?





- Compositions with number of parts divisible by first part have g.f. $\left(1+\frac{z^2}{1-2z}\right)R\left(z-z^2\right)$ (A168655). What's going on?