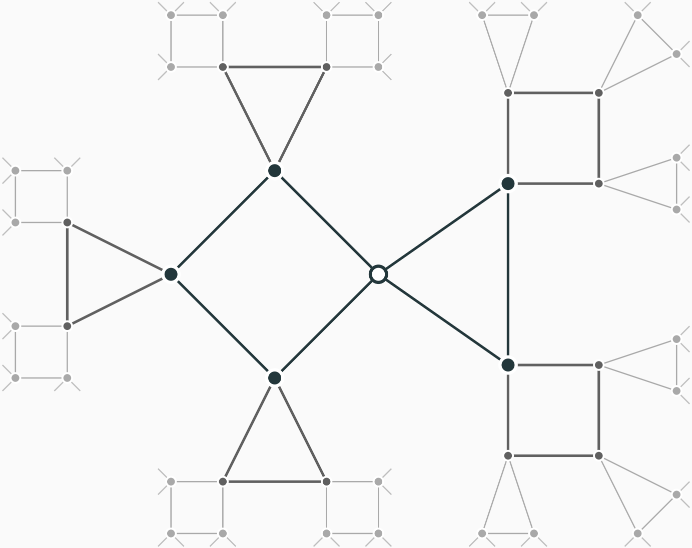
On groups with D-finite cogrowth series

Mudit Aggarwal, UBC Vancouver

Joint work with Murray Elder (UTS) and Andrew Rechnitzer (UBC)

A Picture



Big Picture

A problem from geometric group theory: Cogrowth

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Converting it into a problem on walks on graphs

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Generating functions, analytic combinatorics, pattern theorem, etc.

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Some results about cogrowth via this method

Cogrowth

- Γ is a group, S is a finite generating set, S^{-1} are the inverses
- $W = \{w \in (S \cup S^{-1})^* \ : \ w \equiv e \text{ in } \Gamma\}$
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Cogrowth and the diagonal of the heat kernel on Γ are the same thing, up to rescaling, since

$$\frac{\psi_n}{(2|S|)^n} \ = \ \mathbb{P}\big[\text{SRW returns to } e \text{ at time } n\big].$$

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- Integers and square grid of integers
$$\Psi_{\mathbb{Z}} = \sum_{n \geq 0} \binom{2n}{n} z^{2n} = \frac{1}{\sqrt{1 - 4z^2}}, \qquad \Psi_{\mathbb{Z}^2} = \sum_{n \geq 0} \binom{2n}{n}^2 z^{2n}.$$

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Some results:

- Free group F_k: $\alpha = 2\sqrt{2k-1} < 2k$, non-amenable.
- Abelian / nilpotent / solvable groups are amenable.
- Thompson's group: very unknown amenability.

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($\Leftarrow$): Known via a combination of results of Muller–Schupp and Chomsky–Schützenberger.
($\Rightarrow$): Open.

Our main result

Consider the family:

$$\mathcal{G}(p_1, \ldots, p_k) = \mathcal{G}_k = \left\langle\, a_1, \ldots, a_k \;\middle|\; a_1^{p_1} = a_2^{p_2} = \cdots = a_k^{p_k} \,\right\rangle, \qquad k, p_i \geq 2.$$

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Main theorem

For all $k \geq 2$ and $p_1, \ldots, p_k \geq 2$, the cogrowth series of the above presentations is D-finite but not algebraic. Additionally, the cogrowth (rate) is an algebraic number.

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Main theorem

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- Gives more "explicit" examples of presentations with D-finite cogrowth.
- Supports the longstanding conjecture.

The result and the conjecture

Cogrowth series of $\mathcal{G}_k = \left\langle\, a_1, \ldots, a_k \;\middle|\; a_1^{p_1} = a_2^{p_2} = \cdots = a_k^{p_k} \,\right\rangle$ is not algebraic.

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$$\mathcal{G}_k \equiv \mathbb{Z} \times \left\langle\, a_1, \ldots, a_k \;\middle|\; a_1^{p_1} = a_2^{p_2} = \cdots = a_k^{p_k} \boldsymbol{= e} \,\right\rangle$$

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Even with such a “tame” extension of a virtually-free group, the cogrowth series is not algebraic. Lends more credence to the conjecture.

Strategy

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(1) A normal form.

$\Delta = a_1^{p_1} \equiv a_2^{p_2} \equiv \cdots \equiv a_k^{p_k}$ is central. So $\langle \Delta \rangle$ is a normal subgroup of $\mathcal{G}_k$, and consequently every $w \in \mathcal{G}_k$ has a normal form

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$$F_k(z;q) \;=\; \sum_{n,m} f_{n,m}\, z^n q^m, \qquad f_{n,m} = \#\{w \in (S \cup S^{-1})^* : |w| = n,\; w \equiv \Delta^m\}.$$

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(4) Result using the asymptotics of F_k.
$[q^0]\, F_k(z;q) \sim C\rho^{-n} n^{-2}$ (or n^{-1}) asymptotic is incompatible with algebraicity.

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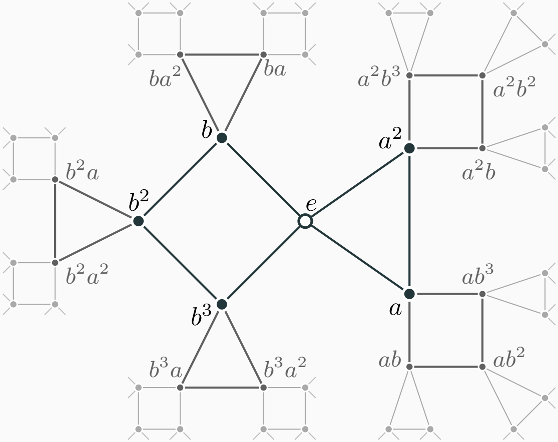

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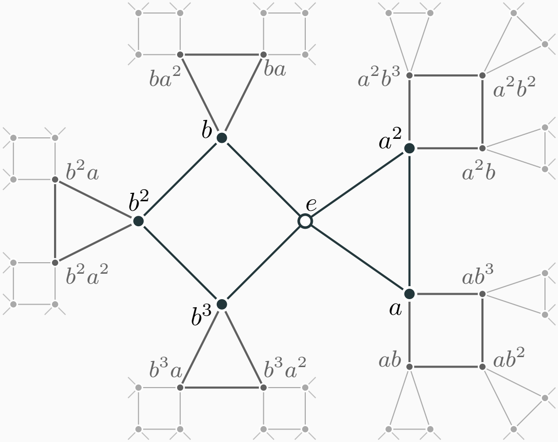

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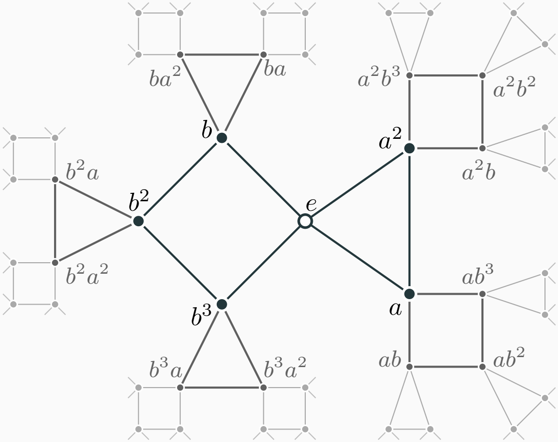

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- Every vertex is labelled by the suffix in the corresponding normal form.
- Vertex-transitive and finite tree-width.
- Every closed walk at the root is in bijection with a word in $\langle \Delta \rangle$.

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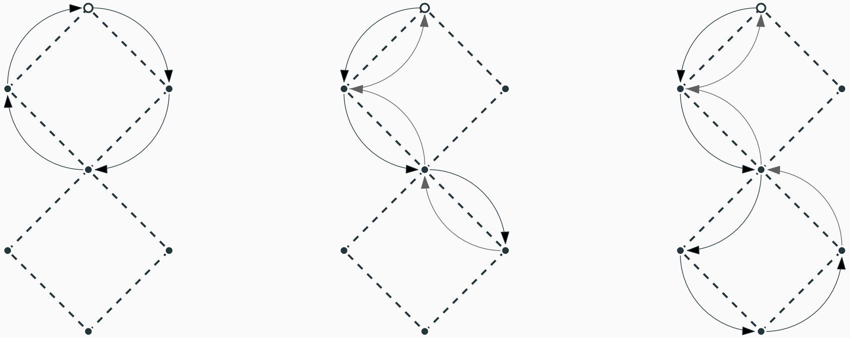

Winding numbers $-1,\ 0,\ +1.$

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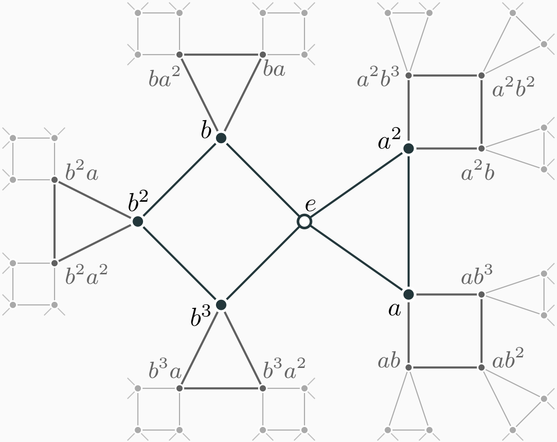

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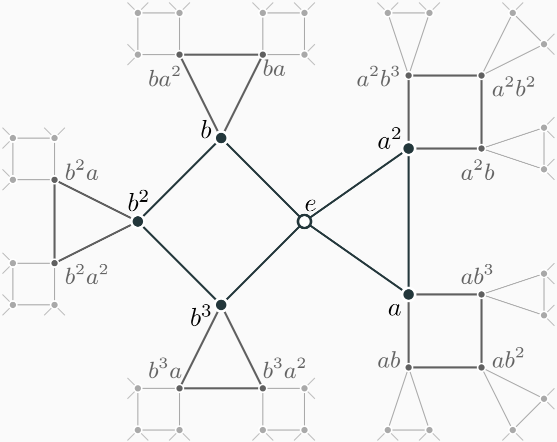

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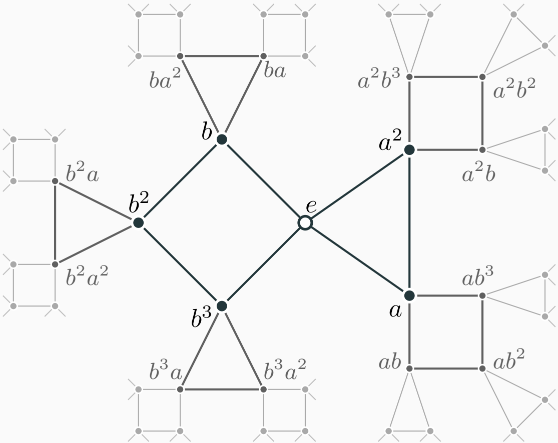

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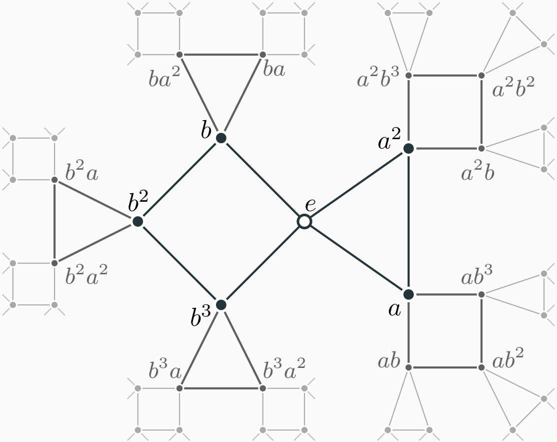

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- As before, vertex labels are in bijection with suffixes of normal forms.
- So $F_k(z; q)$ is *also* the bivariate generating function counting closed walks by length and winding number.

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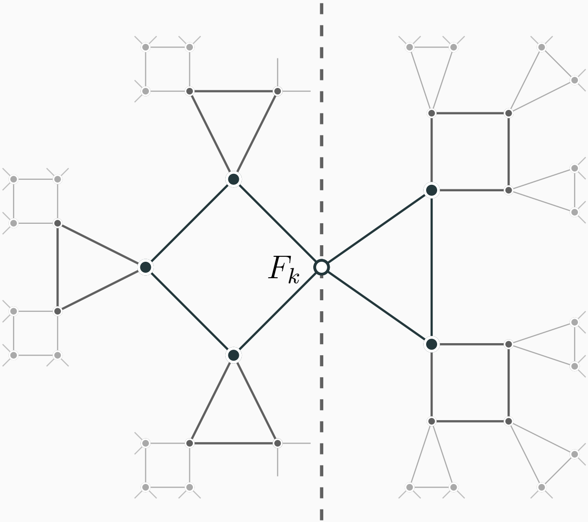

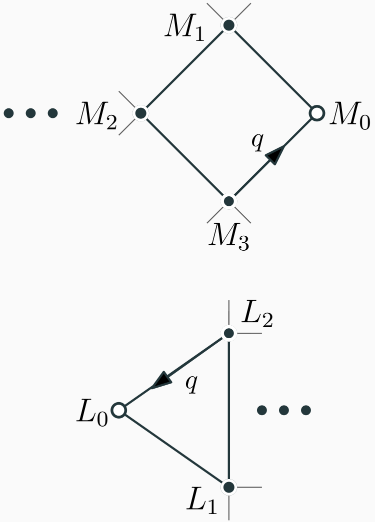

The system of equations

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\begin{aligned}
L_0 &= 1 + z[L_1 + q \cdot L_2], \\
L_1 &= z[L_0 + L_2] \cdot M_0, \\
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Glue facets back together through primitive walks:

$$L_0 = \frac{1}{1 - P_1}, \qquad M_0 = \frac{1}{1 - P_2}, \qquad F_k = \frac{1}{1 - (P_1 + P_2)}.$$

The system of equations, more generally

For $\mathcal{G}(p_1, \ldots, p_k)$, the Schreier graph has k facets intersecting at every vertex, with side lengths $p_1, \ldots, p_k$. For every facet, the system we get is $(1 \le i \le k)$:

$$
\begin{aligned}
L_0^{(i)} &= 1 + z\big[L_1^{(i)} + q\, L_{p_i-1}^{(i)}\big], \\
L_r^{(i)} &= z\big[L_{r-1}^{(i)} + L_{r+1}^{(i)}\big]\sum_{j \ne i} L_0^{(j)} && (1 \le r \le p_i - 2), \\
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- Has nice properties like positive, polynomial, strongly-connected, etc.
- Would like to show a Gaussian local limit law, using some variant of the Drmota-Lalley-Woods theorem.

Variance

Want to show

$$[z^n q^m]L_0^{(i)} = c_{i,2n,2m} = \frac{a_i \rho^{-2n}}{n^2}\left(\exp\left(\frac{-b_i m^2}{\sigma^2}\right) + \mathcal{O}(\sqrt{n})\right)$$

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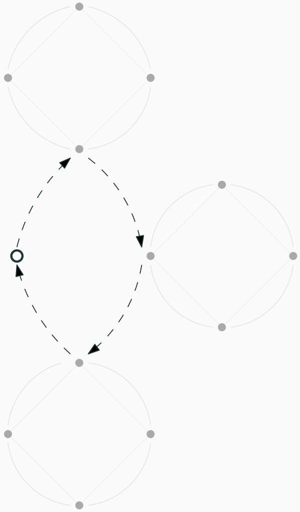
- We deal with it by proving a *pattern theorem* for walks.
- A pattern theorem says that most large members of a combinatorial class have a positive density of any fixed local pattern or motif.
- The first pattern theorem is attributed to Kesten (1963) for self-avoiding walks.

Pattern Theorem



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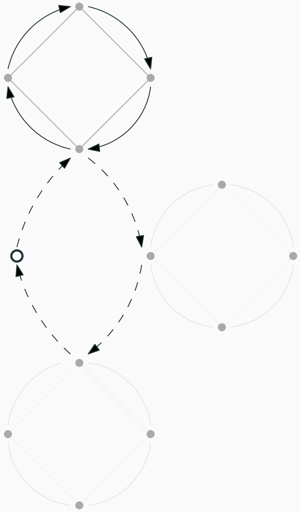

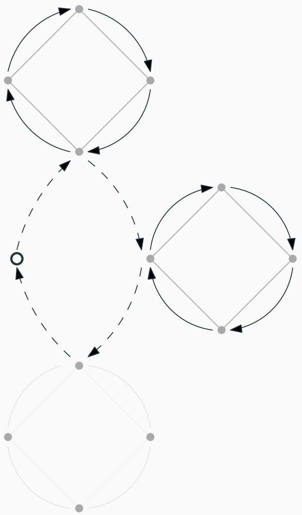

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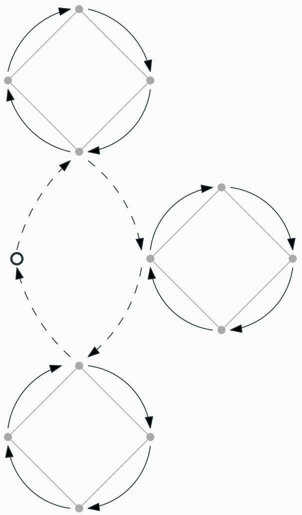

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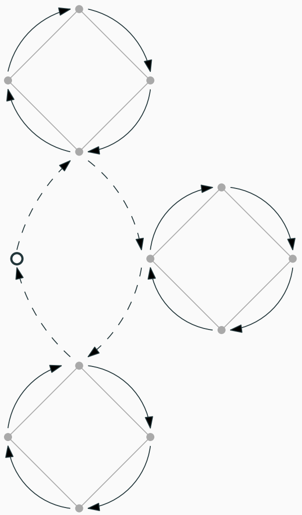
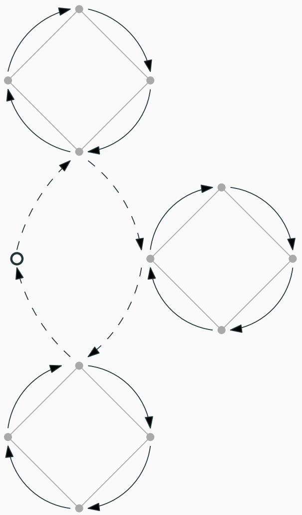


- Consider a *pattern* to be a single loop along a facet.

Pattern Theorem



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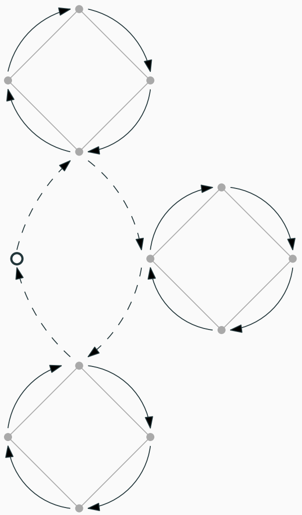

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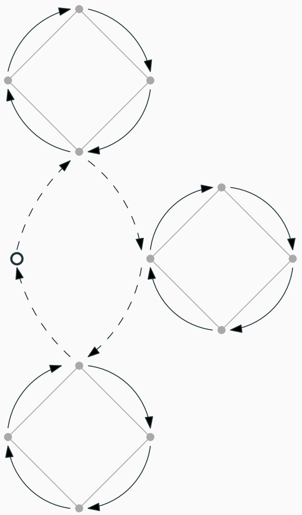

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- Since these loops can be independently reversed, the variance is $\Omega(n)$.

Partial Asymptotics

One can put all this together to get the asymptotics for walks on partial graphs.

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Theorem 1

Let $L_0^{(i)} = \sum_{n,m} c_{i,n,m} z^n q^m$. Then, asymptotically

$$c_{i,2n,2m} = \frac{a_i \rho^{-2n}}{n^2} \left(\exp\left(\frac{-m^2}{\gamma n} \right) + \mathcal{O}(\sqrt{n}) \right),$$

when m is close to 0, for some positive constants a_i, ρ, γ.

From L_0 to F_k

$$L_0^{(i)} = \frac{1}{1 - P_i}, \qquad F_k = \frac{1}{1 - \sum_i P_i}.$$

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- Using that, one can show that the expected number of returns is $\mathcal{O}(1)$.

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Theorem 2

Let $F_k = \sum_{n,m} f_{n,m} z^n q^m$. Then, asymptotically

$$f_{2n,2m} = \frac{C\rho^{-2n}}{n^2}\left(\exp\left(\frac{-m^2}{\gamma n}\right) + \mathcal{O}(\sqrt{n})\right),$$

when m is close to 0, for some positive constants C, ρ, γ.

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Consequently, $\Psi_{\mathcal{G}_k} = [q^0]F_k$ is D-finite but not algebraic due to the n^{-2} polynomial factor.

fin.

Thank you!