Even-Odd Partition Identities of Göllnitz–Gordon Type

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Given integers $n \geqslant 0$ and $i \in \{1, 2\}$, let

$$\mathcal{G}_i(n) = \left\{ \lambda \vdash n : \begin{array}{l} \text{each part of } \lambda \ \geqslant 2i-1, \\ \lambda \text{ has no equal or consecutive parts,} \\ \lambda \text{ has no consecutive even parts} \end{array} \right\},$$

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Then $|\mathcal{G}_i(n)| = |\mathcal{G}'_i(n)|$.

Example: $n = 10$

$\mathcal{G}_1(10)$	$\mathcal{G}_1'(10)$
10	
$9+1$	
$8+2$	
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$7+3$	$4+1+1+1+1+1+1$
$6+3+1$	$1+1+1+1+1+1+1+1+1+1$

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Given integers $n \geqslant 0$ and $i \in \{1, 2\}$, let

$$\mathcal{H}_i(n) = \left\{ \lambda \vdash n : \begin{array}{l} \text{odd parts are distinct and } \geqslant 2i-1, \\ \text{even parts are } \geqslant 2(\ell(\lambda)+i-1) \end{array} \right\}.$$

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$9+1$	$7+1+1+1$	$9+1$
$8+2$	$4+4+1+1$	$7+3$
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First proof: Recurrence approach

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- Define

$$H_i(r,s,n) = \#\left\{\lambda \in \mathcal{H}_i(n) : \ell(\lambda^{(e)}) = r,\ \ell(\lambda^{(o)}) = s\right\},$$

A new system of recursion formulas:

$$H_i(r,s,n) = \begin{cases} 1, & \text{if } (r,s,n) = (0,0,0), \\ 0, & \text{if } \min\{r,s,n\} \leqslant 0,\ (r,s,n) \neq (0,0,0). \end{cases}$$

$$\begin{aligned} H_1(r,s,n) - H_2(r,s,n) = {} & H_1(r,s-1,n-2(r+s)+1) \\ & + H_1(r-1,s,n-4(r+s)+2). \end{aligned}$$

$$H_2(r,s,n) = H_1(r,s,n-2(r+s)).$$

Second equation of the system

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Second proof: Generating series

Göllnitz–Gordon identities: analytic version

For $i \in \{1, 2\}$ we have:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+(2i-2)n}(-q, q^2)_n}{(q^2, q^2)_n} = \frac{1}{(q, q^8)_\infty (q^{2i-1}, q^8)_\infty (q^{9-2i}, q^8)_\infty}.$$

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- q-Pochhammer symbol :

$$(a;q)_n = \begin{cases} 1, & n=0, \\ \prod_{k=0}^{n-1}(1-aq^k), & n \geqslant 1, \end{cases} \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Theorem (P.A., 2026)

For $i \in \{1, 2\}$ we have:

$$\sum_{n=0}^{\infty} H_i(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2+(2i-2)n}(-q, q^2)_n}{(q^2, q^2)_n},$$

where $H_i(n) = |\mathcal{H}_i(n)|$.

Sketch of the proof (for $i = 1$)

- $\mathcal{H}_1(n) = \left\{ \lambda \vdash n : \begin{array}{l} \text{odd parts are distinct,} \\ \text{even parts } \geqslant 2\ell(\lambda) \end{array} \right\}.$

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- $$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_r \, (q;q)_{n-r}}, \qquad 0 \leqslant k \leqslant n.$$

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- The finite q-binomial theorem:

$$(-z,q)_n = \prod_{r=0}^{n-1} (1 + zq^r) = \sum_{r=0}^{n} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q z^r.$$

Sketch of the proof (for $i = 1$)

$$\sum_{n=0}^{\infty} H_1(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2+(2i-2)n}(-q,q^2)_n}{(q^2,q^2)_n} = \sum_{n=0}^{\infty} G_1(n) q^n.$$

¡Thank you for your attention!