e-positivity of integral form Macdonald polynomials

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Macdonald polynomials

[Selberg's integral] For $k, a, b \in \mathbb{C}$,

$$\int_{(0,1)^n} \Big|\prod_{1\le i<j\le n}(x_i-x_j)\Big|^{2k}\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}dx_1\cdots dx_n = \prod_{i=1}^{n}\frac{\Gamma(a+(i-1)k)\Gamma(b+(i-1)k)\Gamma(ik+1)}{\Gamma(a+b+(n+i-2)k)\Gamma(k+1)}.$$

- Kadell (1988) conjectured the existence of symmetric functions which could be inserted into the integrand to generalize Selberg's integral in a certain way.
- Macdonald (1988) showed how to define such, by means of orthogonality with respect to a generalization of the Hall scalar product.

[Macdonald's generalization] For $t = q^k$ for some $k \in \mathbb{N}$,

$$\frac{1}{n!}\int_{(0,1)^n}\mathbf{P}_\lambda(\mathbf{X};\mathbf{q},\mathbf{t})\prod_{1\le i<j\le n}\prod_{r=0}^{k-1}(x_i-q^r x_j)(x_i-q^{-r}x_j)\prod_{i=1}^{n}x_i^{a-1}(x_i;q)_{b-1}d_qx_1\cdots d_qx_n$$

$$= q^F\prod_{i=1}^{n}\frac{\Gamma_q(\lambda_i+a+(i-1)k)\Gamma_q(b+(i-1)k)}{\Gamma_q(\lambda_i+a+b+(n+i-2)k)}\prod_{1\le i<j\le n}\frac{\Gamma_q(\lambda_i-\lambda_j+(j-i+1)k)}{\Gamma_q(\lambda_i-\lambda_j+(j-i)k)}.$$

- The **integral form** Macdonald polynomial $J_\mu(X;q,t)$ is defined by

$$J_\mu(X;q,t) = \prod_{c\in\mu}(1-q^{\mathrm{arm}(c)}t^{\mathrm{leg}(c)+1})P_\mu(X;q,t).$$

Motivation

- The **integral form** Macdonald polynomial $J_\mu(X;q,t)$ satisfies

$$J_\mu(X;q,t) = \sum_{\lambda\vdash n}K_{\lambda\mu}(q,t)\,s_\lambda[X(1-t)].$$

Then the **modified** Macdonald polynomial is defined by

$$\tilde{H}_\mu(X;q,t) = t^{n(\mu)}J_\mu\left[\frac{X}{1-1/t};q,1/t\right] = \sum_{\lambda\vdash n}\tilde{K}_{\lambda\mu}(q,t)\,s_\lambda(X).$$

- By the Carlsson–Mellit relation, we have

$$\chi_\pi(X;q) = \frac{\mathrm{LLT}_\pi[(q-1)X;q]}{(q-1)^n}.$$

Schur-positive		e-positive ?
$t^{-\sum_i i\mu_i}\tilde{H}_\mu[(t-1)X;q,t]$	$=$	$J_\mu(X;q,t^{-1})$
$\dfrac{\mathrm{LLT}_\gamma[(q-1)X;q]}{(q-1)^n}$	$=$	$\chi_\gamma(X;q)$

Conjectures

- Haglund's conjecture:

$$\left\langle \frac{J_\mu(X;q,q^\alpha)}{(1-q)^n}, s_\lambda(X)\right\rangle \in \mathbb{N}[q]$$

- Haglund–Wilson conjecture: for any $\alpha\in\mathbb{N}$,

$$\left\langle \frac{J_\mu(X;q,q^{-\alpha})}{(q-1)^n}, s_\lambda(X)\right\rangle \in \mathbb{N}[q]$$

after some negative q-factors taken care of.
- Jaeseong's conjecture:

$$q^{\alpha\sum_i i\cdot\mu_i}\frac{J_\mu(X;q,q^{-\alpha})}{(q-1)^n} \text{ is } e\text{-positive.}$$

- Alexandersson–Haglund–Wang conjecture:

$$\left\langle \tilde{J}_\mu^{(\alpha)}(X), s_\lambda(X)\right\rangle = \sum_{k=0}^{n-1}a_k(\mu,\lambda)\binom{\alpha+k}{n} = \sum_{k=1}^{n}b_{n-k}(\mu,\lambda)\binom{\alpha}{k}k!,$$

with $a_k(\mu,\lambda)$, $b_{n-k}(\mu,\lambda)\in\mathbb{N}$.

e-expansion of $J_{\mathbf{hook}}(X;q,t)$

Let $\nu\vdash n$.

- A *ribbon tiling* of ν is a tiling of ν with ribbons. The *type* of a ribbon tiling is the partition $\lambda = (1^{m_1}, 2^{m_2}, \dots)$, where m_i is the number of ribbons of size i.
- Let $\mathrm{RT}(\nu)$ denote *the set of ribbon tilings* of ν. We also define $\mathrm{RT}(\nu,\lambda)$ to be the set of ribbon tilings of ν with type λ.
- A *special ribbon tiling* of ν is a ribbon tiling of ν such that every ribbon has at least one cell in the first column of ν.
- Let $\mathrm{SRT}(\nu)$ denote *the set of special ribbon tilings* of ν, and let $\mathrm{SRT}(\nu,\lambda)$ denote the set of special ribbon tilings of ν with type λ.

Definition. Let $\nu\vdash n$ and $T\in\mathrm{RT}(\nu)$. Define

$$\mathrm{wt}(T) = \prod_{c\in\nu}\mathrm{wt}(c,T,\nu),$$

where

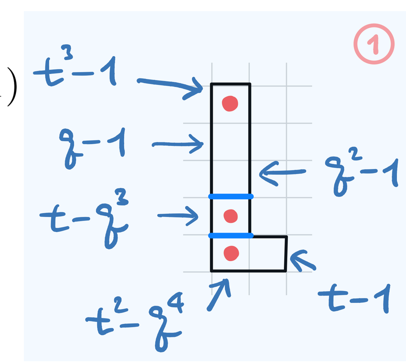
$$\mathrm{wt}(c,T,\nu) = \begin{cases} 1-q^{\mathrm{leg}_\nu(c)}t^{|\theta|} & \text{if } c \text{ is the head of a ribbon } \theta \text{ in } T,\\ q^{\mathrm{leg}_\nu(c)}-t^{\mathrm{coarm}_\nu(c)} & \text{otherwise.}\end{cases}$$

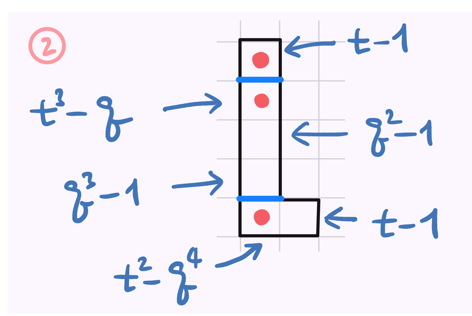
Theorem. Let $\mu=(r,1^s)$, where $r+s=n$. Then

$$J_\mu(X;q,t) = \sum_{\lambda\ge\mu'}\left(\sum_{T\in\mathrm{SRT}(\mu',\lambda)}\mathrm{wt}(T)\right)e_\lambda.$$

Examples

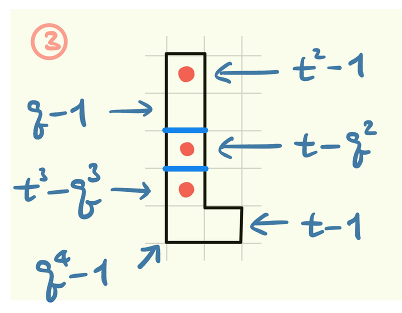
$$[e_{321}]J_{(5,1)}(X;q,t^{-1}) = t^{-7}\big((t^3-1)(q-1)(q^2-1)(t-q^3)(t^2-q^4)(t-1)$$

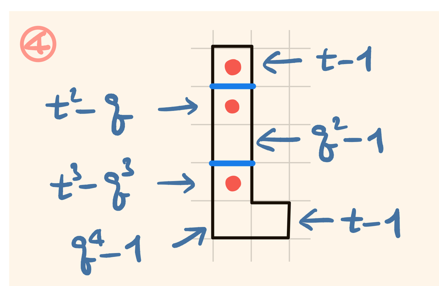




$$+(t-1)(t^3-q)(q^2-1)(q^3-1)(t^2-q^4)(t-1)$$

$$+(t^2-1)(q-1)(t-q^2)(t^3-q^3)(q^4-1)(t-1)$$





$$+(t-1)(t^2-q)(q^2-1)(t^3-q^3)(q^4-1)(t-1)\big).$$

Separately, we have e-expansion of $J_{2^\ell1^{n-2\ell}}(X;q,t)$.

Theorem.

$$[e_{(n-k,k)}]J_{2^\ell1^{n-2\ell}}(x;q,t) = q^{\ell-k}(1-t^{n-2k})(q^{-1};t)_{\ell-k}(t;t)_{n-\ell-k-1}(t;t^{\ell-k+1})_k(qt^{n-\ell-k+1};t)_k$$
$$= q^{\ell-k}(1-t^{n-2k})(q^{-1};t)_{\ell-k}(t;t)_{n-\ell-k-1}\frac{(t;t)_\ell}{(t;t)_{\ell-k}}\frac{(qt;t)_{n-\ell}}{(qt;t)_{n-\ell-k}}.$$

We also computed e_n-coefficient of $J_\mu(X;q,t)$ for any $\mu\vdash n$.

Theorem.

$$[e_n]J_\mu(x;q,t) = (1-t^n)\prod_{c\in\mu'\setminus\{(\mu_1,1)\}}(q^{\mathrm{coleg}(c)}-t^{\mathrm{coarm}(c)}).$$

α-chromatic symmetric functions

There are two families of symmetric functions indexed by the set of Dyck paths:

- the **chromatic quasisymmetric functions**

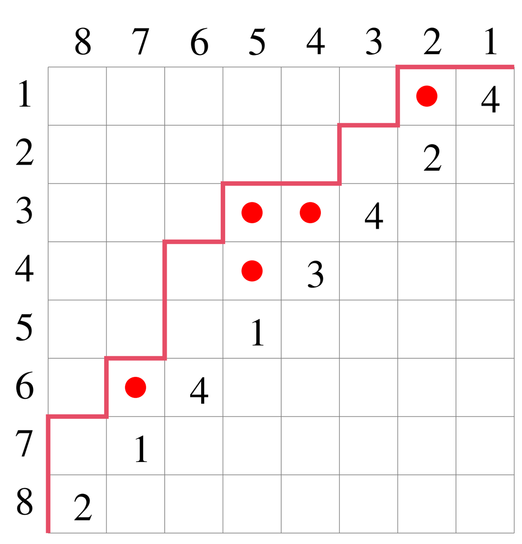
$$\chi_\pi(X;q) = \sum_{\substack{w\in\mathbb{Z}^n_{>0}\\ \text{proper}}}q^{\mathrm{inv}_\pi(w)}x_1^{w(1)}x_2^{w(2)}\cdots x_n^{w(n)},$$

- the **unicellular LLT polynomials**

$$\mathrm{LLT}_\pi(X;q) = \sum_{w\in\mathbb{Z}^n_{>0}}q^{\mathrm{inv}_\pi(w)}x_1^{w(1)}x_2^{w(2)}\cdots x_n^{w(n)},$$

where

$$\mathrm{inv}_\pi(w) = |\{(i,j) : (i,j)\in\mathrm{Area}(\pi),\ w(i)<w(j)\}|.$$



$q^5x_1^2x_2^2x_3x_4^3$

Theorem. [Carlsson–Mellit, 2018]

$$\chi_\pi(X;q) = \frac{\mathrm{LLT}_\pi[(q-1)X;q]}{(q-1)^n}.$$

Definition. **[α-chromatic symmetric functions]**

$$\chi_\pi^{(\alpha)}(X;q) = \frac{\mathrm{LLT}_\pi[(q^\alpha-1)X;q]}{(q-1)^n} = \chi_\pi\left[\frac{q^\alpha-1}{q-1}X;q\right].$$

Relation to α-chromatic symmetric functions

Griffin et al.[2] expanded the unicellular LLT polynomials in terms of modified Macdonald polynomials

$$\mathrm{LLT}_{\mathbf{e}}(X;q) = \sum_\mu C_{\mathbf{e},\mu}(q,t)\,\widetilde{\mathrm{H}}_\mu(X;q,t).$$

By using several relations of Macdonald polynomials of different forms, we get

$$\mathrm{LLT}_{\mathbf{e}}[(1-t)X;q] = \sum_\mu C_{\mathbf{e},\mu}(q,t^{-1})t^{-n(\mu)}J_\mu(X;q,t).$$

If we let $t=q^{-k}$, for any positive integer k, and divide both sides by $(q-1)^n$, then we get

$$\chi_{\mathbf{e}}^{(k)}(X;q) = \frac{\mathrm{LLT}_{\mathbf{e}}[(q^k-1)X;q]}{(q-1)^n} = \sum_\mu C_{\mathbf{e},\mu}(q,q^k)q^{k\sum i\mu_i}\frac{J_\mu(X;q,q^{-k})}{(q-1)^n}.$$

References

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