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Immanant Positivity for Catalan–Stieltjes Matrices

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Catalan–Stieltjes matrices

- $\gamma = (r_k)_{k\geq 0}$, $\sigma = (s_k)_{k\geq 0}$, $\tau = (t_k)_{k\geq 1}$.
- Catalan–Stieltjes matrix $C^{\gamma,\sigma,\tau} = (c_{n,k})_{n,k\geq 0}$:

$$c_{0,0} = 1$$
$$c_{n,0} = s_0 c_{n-1,0} + t_1 c_{n-1,1},$$
$$c_{n,k} = r_{k-1}c_{n-1,k-1} + s_k c_{n-1,k} + t_{k+1}c_{n-1,k+1},$$

- The associated Hankel matrix:

$$H^{\gamma,\sigma,\tau} = (c_{i+j,0})_{i,j\geq 0}.$$

- The entries $c_{n,0}$: Catalan-like numbers.

Immanants

- For a matrix $M = (m_{ij})_{1\leq i,j\leq n}$ and $\lambda \vdash n$, define

$$\mathrm{Imm}_\lambda M = \sum_{\pi\in\mathfrak{S}_n} \chi^\lambda(\pi)\prod_{i=1}^n m_{i,\pi(i)}$$

to be the immanant of M with respect to λ.
- This concept is a generalization of the determinant because $\mathrm{Imm}_{(1^n)} M = \det M$.

q-positivity

- $f(q)$ is called q-nonnegative if $f(q) \in \mathbb{N}[q]$, written as $f(q) \geq_q 0$.
- A matrix M is called q-totally positive (q-TP) if all its minors are q-nonnegative.

Main result

If $\gamma = (r_k)_{k\geq 0}$, $\sigma = (s_k)_{k\geq 0}$ and $\tau = (t_k)_{k\geq 1}$ are q-nonnegative polynomials and satisfy one of the following conditions:

(1) $s_0 \geq_q r_0$, and $s_k \geq_q r_k + t_k$ for $k \geq 1$;
(2) $s_0 \geq_q t_1$, and $s_k \geq_q r_{k-1} + t_{k+1}$ for $k \geq 1$;
(3) $s_0 \geq_q 1$, and $s_k \geq_q r_{k-1}t_k + 1$ for $k \geq 1$;
(4) $s_0 \geq_q r_0 t_1$, and $s_k \geq_q r_k t_{k+1} + 1$ for $k \geq 1$;
(5) there exist two q-nonnegative polynomials b_k and c_k such that $r_k = 1$, $s_k = b_k + c_k$, and $t_{k+1} = b_{k+1}c_k$ for each $k \geq 0$,

Then every immanant of every square submatrix of $C^{\gamma,\sigma,\tau}$ or $H^{\gamma,\sigma,\tau}$ is q-nonnegative.

A stronger form

Let γ, σ, τ be given as above. Then for any $n\times n$ square submatrix M of $C^{\gamma,\sigma,\tau}$ or $H^{\gamma,\sigma,\tau}$ and any irreducible character χ^λ, we have

$$\mathrm{Imm}_\lambda M - \deg(\chi^\lambda)\det M \geq_q 0.$$

Motivation

- Pan and Zeng (2016), Wang and Zhu (2016): $C^{\gamma,\sigma,\tau}$ and $H^{\gamma,\sigma,\tau}$ are q-TP.
- Stembridge (1991): $\mathrm{Imm}_\lambda M \geq 0$ for TP matrices.
- Our result is a generalization of the above two results.

Proof sketch

Finding recursive formulas of $C^{\gamma,\sigma,\tau}$ and $H^{\gamma,\sigma,\tau}$

↓

Construct planar networks for $C^{\gamma,\sigma,\tau}$ and $H^{\gamma,\sigma,\tau}$

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Deduce immanant positivity by means of Goulden-Jackson, Greene, Stembridge and Wolfgang's results.

Recurrence formula for $C^{\gamma,\sigma,\tau}$

- $C_n = (c_{i,j})_{0\leq i,j\leq n}$: the nth leading principal submatrix of $C^{\gamma,\sigma,\tau}$.
- The recurrence relation can be written as

$$C_{n+1} = \bar{C}_n L_n, \qquad \bar{C}_n = \begin{pmatrix} 1 & O \\ O & C_n \end{pmatrix},$$

where

$$L_n = \begin{pmatrix} 1 & & & & \\ s_0 & r_0 & & & \\ t_1 & s_1 & r_1 & & \\ & \ddots & \ddots & \ddots & \\ & & t_n & s_n & r_n \end{pmatrix}.$$

Digraph construction for L_n

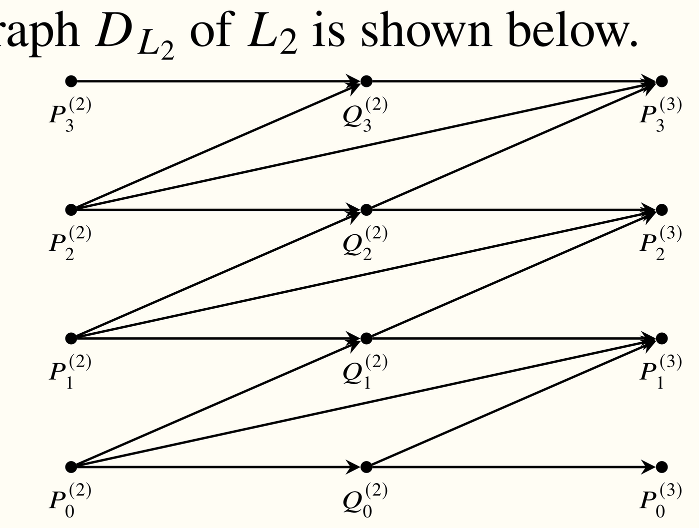
- Set $P_i^{(n)} = (2n, i)$, $Q_i^{(n)} = (2n+1, i)$.
- The vertex set:

$$P_0^{(n)}, \ldots, P_{n+1}^{(n)}, Q_0^{(n)}, \ldots, Q_{n+1}^{(n)}, P_0^{(n+1)}, \ldots, P_{n+1}^{(n+1)}.$$

- The arcs:
  (1) horizontal: $P_k^{(n)} \to Q_k^{(n)}$ and $Q_k^{(n)} \to P_k^{(n+1)}$ for $0 \leq k \leq n+1$;
  (2) diagonal: $P_k^{(n)} \to Q_{k+1}^{(n)}$ and $Q_k^{(n)} \to P_{k+1}^{(n+1)}$ for $0 \leq k \leq n$;
  (3) super-diagonal: $P_k^{(n)} \to P_{k+1}^{(n+1)}$ for $0 \leq k \leq n$.

An example

The digraph D_{L_2} of L_2 is shown below.



Planar network for L_n

- Now we give the weights of the arcs. In case (1),

$$\mathrm{wt}(P_k^{(n)} \to Q_k^{(n)}) = r_{n-k},$$
$$\mathrm{wt}(P_k^{(n)} \to Q_{k+1}^{(n)}) = t_{n-k},$$
$$\mathrm{wt}(P_k^{(n)} \to P_{k+1}^{(n+1)}) = s_{n-k} - r_{n-k} - t_{n-k},$$

and the remaining arcs all have weight 1.
- The other four cases are similar and the weights correspond to the conditions.
- Under this construction, the (i,j)-entry of L_n is exactly the sum of weights of all directed paths from $P_{n+1-i}^{(n)}$ to $P_{n+1-j}^{(n+1)}$, that is,

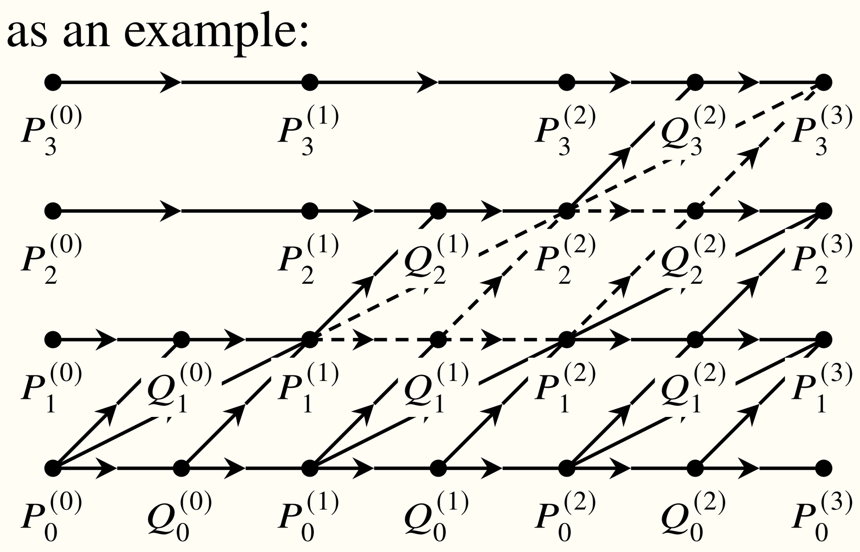
$$L_n = \left(\mathrm{GF}_{\mathcal{L}_n}\left(P_{n+1-i}^{(n)}, P_{n+1-j}^{(n+1)}\right)\right)_{0\leq i,j\leq n+1},$$

Network for C_n

- The planar network for C_n can be recursively constructed from that of $L_0, L_1, \ldots, L_{n-1}$.
- The resulting planar network satisfies

$$C_n = \left(\mathrm{GF}_{\mathcal{C}_n}\left(P_{n-i}^{(0)}, P_{n-j}^{(n)}\right)\right)_{0\leq i,j\leq n}.$$

- C_3 as an example:



Network for $H^{\gamma,\sigma,\tau}$

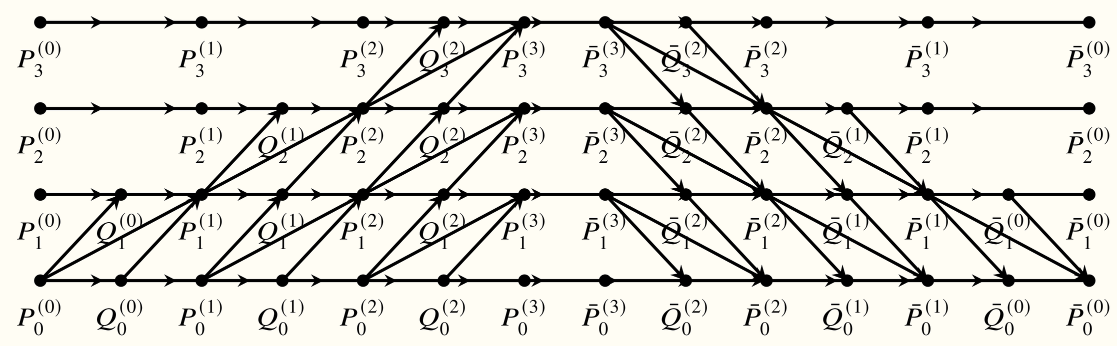
- Fix $k \geq 0$. Let H_n be the nth leading principal submatrix of $H^{\gamma,\sigma,\tau}$. Then

$$H_n = \left(\mathrm{GF}_{\mathcal{C}_{2n+k}}\left(P_{n+k-i}^{(n+k-i)}, \bar{P}_{n+k+j}^{(n+k+j)}\right)\right)_{0\leq i,j\leq n}.$$

- When $r_k = 1$ for $k \geq 0$, we have

$$H_n = C_n T_n C_n^T, T_n = \mathrm{diag}(1, t_1, t_1t_2, \ldots, t_1t_2\cdots t_n).$$

H_3 as an example for another construction in this special case:



From networks to positivity

- Goulden and Jackson (1992), Greene (1992), Stembridge (1991), Wolfgang (1997):
- *If a planar network for a matrix M is D-compatible, then the immanants of all square submatrices of M are nonnegative polynomials in the arc weights.*
- Applying this theorem to our planar networks yields the main result since in the construction all the arc weights are q-nonnegative.

New inequalities from immanant positivity

- Let $(a_k)_{k\geq 0}$ be a sequence of Catalan-like numbers.
- For any $0 \leq i_1 < i_2 < i_3$ and $0 \leq j_1 < j_2 < j_3$,

$$2(a_{i_1+j_2}a_{i_2+j_1}a_{i_3+j_3} + a_{i_1+j_3}a_{i_2+j_2}a_{i_3+j_1} + a_{i_1+j_1}a_{i_2+j_3}a_{i_3+j_2}) - 3(a_{i_1+j_2}a_{i_2+j_3}a_{i_3+j_1} + a_{i_1+j_3}a_{i_2+j_1}a_{i_3+j_2}) \geq_q 0.$$

- In particular, for $0 \leq i < j < k$,

$$a_{2i}a_{j+k}^2 + a_{2j}a_{i+k}^2 + a_{2k}a_{i+j}^2 - 3a_{i+j}a_{j+k}a_{k+i} \geq_q 0.$$

Applications to traditional combinatorial polynomials

- The Eulerian polynomials $E_n(q)$ are defined by

$$\sum_{m\geq 0}(m+1)^n q^m = \frac{E_n(q)}{(1-q)^{n+1}},$$

and they are generated by $r_k = k+1, s_k = k(q+1)+1, t_{k+1} = (k+1)q$, which satisfy (1).
- Therefore, for $0 \leq i < j < k$,

$$E_{2i}E_{j+k}^2 + E_{2j}E_{i+k}^2 + E_{2k}E_{i+j}^2 \geq_q 3E_{i+j}E_{j+k}E_{k+i}.$$

- The same is true for Schröder polynomials and Narayana polynomials of type A.

Selected references

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