Congruences for hook lengths of partitions

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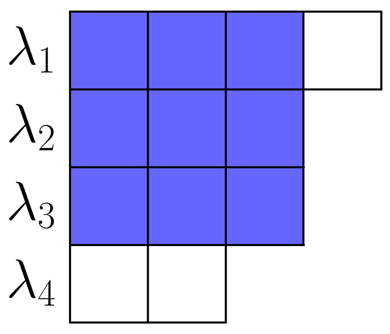
Introduction

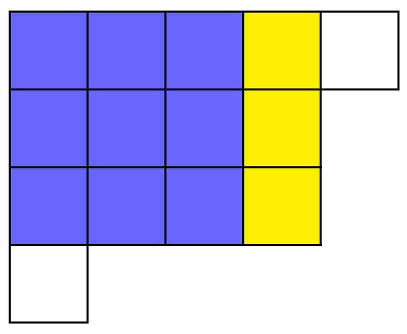
- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n: positive integer sequence such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ and $|\lambda| = \sum_{i=1}^{\ell} \lambda_i = n$. $\mathcal{P} = \{\text{integer partition}\}$.
- Frobenius notation
$$\lambda = \begin{pmatrix} a_1 & a_2 & \ldots & a_d \\ b_1 & b_2 & \ldots & b_d \end{pmatrix}$$
- z-asymmetric partition $\lambda \in \mathcal{P}_z$: partition such that
$$\lambda = \begin{pmatrix} a_1+z & a_2+z & \ldots & a_d+z \\ a_1 & a_2 & \ldots & a_d \end{pmatrix}.$$
- $\mathrm{BG}_{z,t}$: for $t \geq 2$ and $0 \leq z \leq t-1$, set of partitions in $\mathcal{P}_z$ such that in their Frobenius notation, $(a_j + k)/t \notin \mathbb{N}$ and $(2a_j + z + 1)/t \notin 2\mathbb{N}+1$, for all $1 \leq j \leq d$ and $1 \leq k \leq z$.
- A t-core is a partition with no hook length divisible by t, where the hook length is the number $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ for a box $(i,j) \in \lambda$, $\mathcal{H}(\lambda) := \{\text{hook length}\}$. For $t \in \mathbb{N}^*$, $\mathcal{H}_t(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod{t}\}$ is the set of all hook lengths divisible by t.



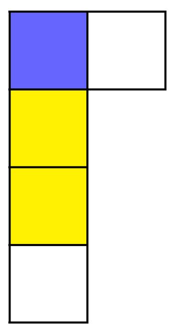
(a) Durfee square

7	6	4	1
5	4	2	
4	3	1	
2	1		

(b) Hook lengths



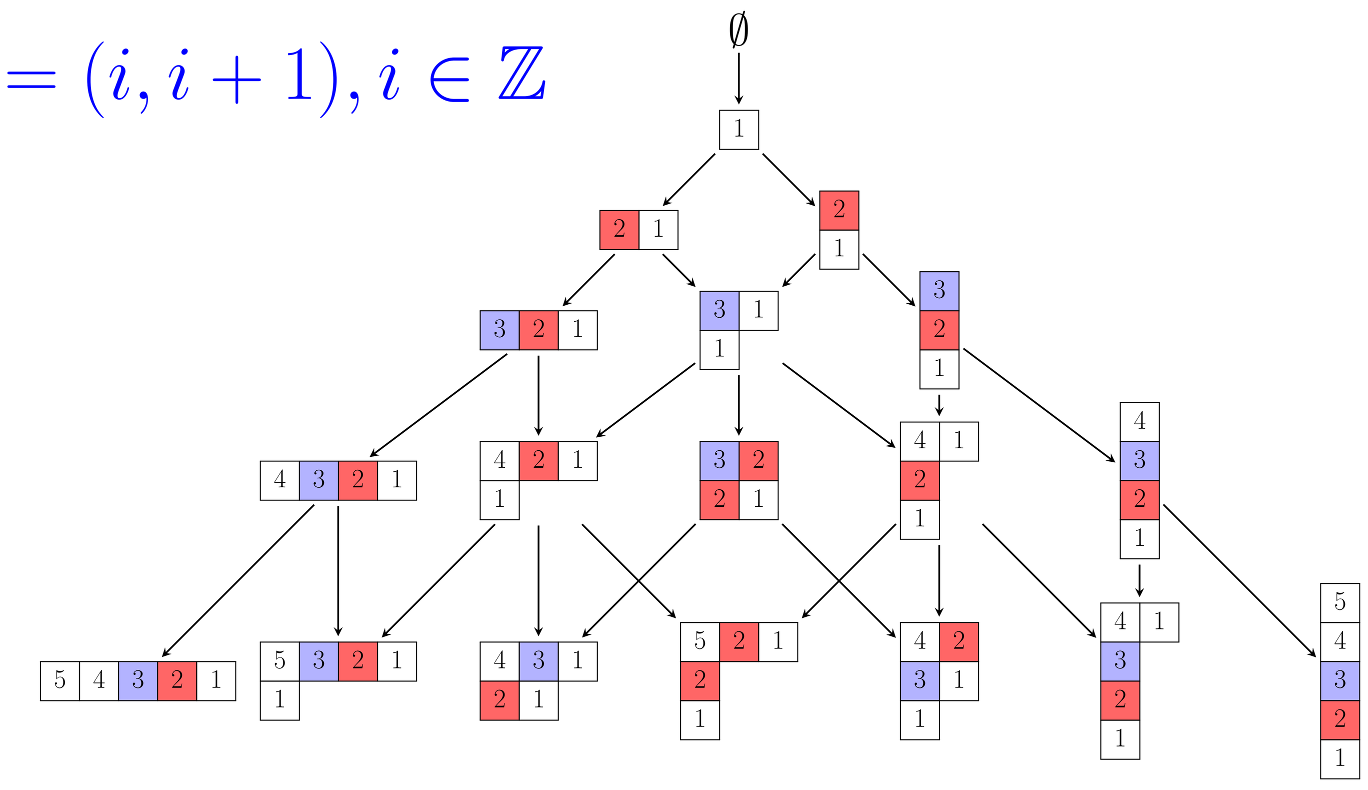
(c) $(5,4,4,1) \in \mathcal{P}_1$



(d) $(2,1,1,1) \in \mathcal{P}_{-2}$

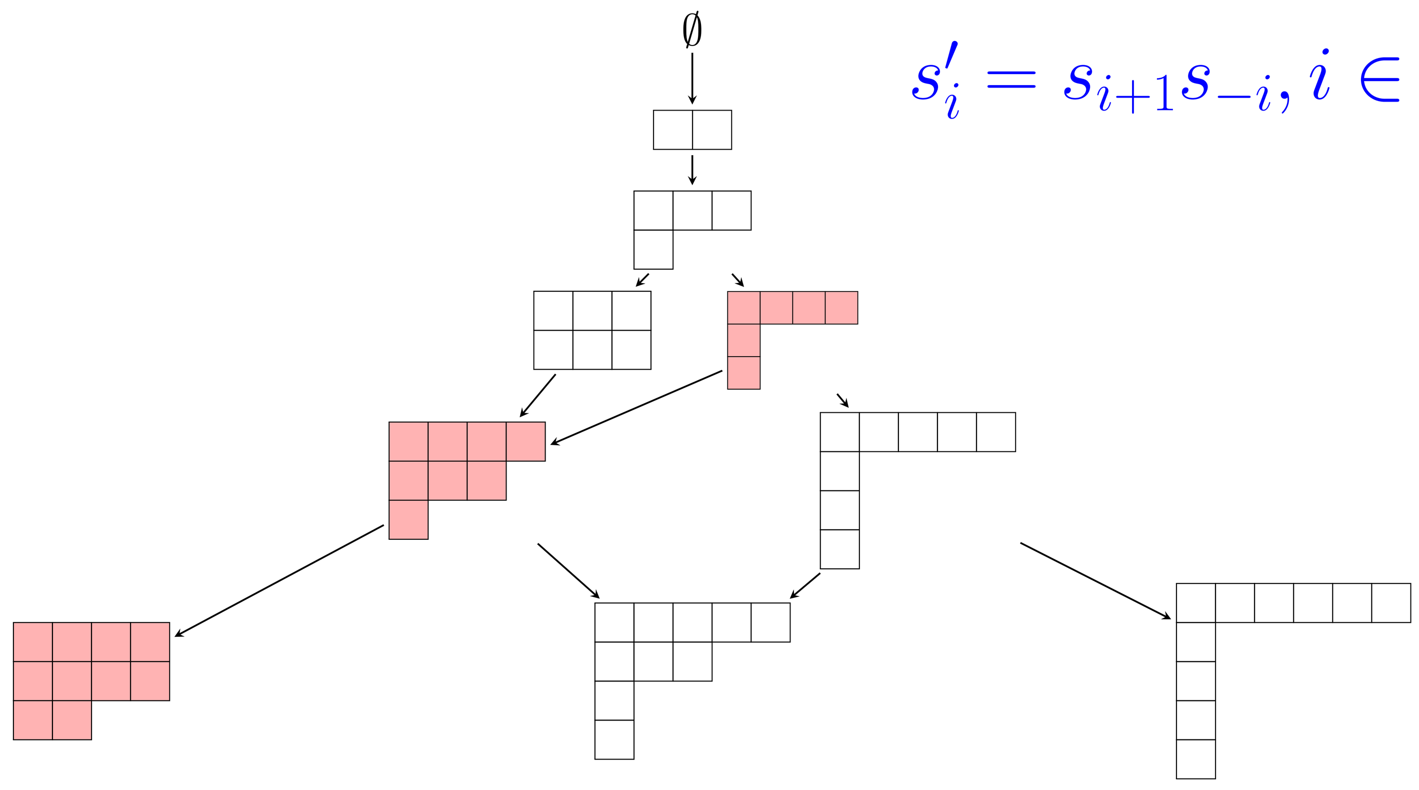
Paths in the Young lattice

$$s_i = (i, i+1), i \in \mathbb{Z}$$



Paths in the $\mathcal{P}_1$ lattice

$$s'_i = s_{i+1}s_{-i}, i \in \mathbb{Z}$$



Main theorem

Let $t \geq 2$ and $0 \leq z \leq t-1$ be integers, and ρ_1, ρ_2 be functions defined on $\mathbb{N}$. Let f_t and g_t be the following formal power series:

$$f_t(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th)^2 \quad \text{and} \quad g_t(q) = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th)^2 \sum_{h \in \mathcal{H}(\lambda)} \rho_2(th)$$

Then we have

$$\sum_{\lambda \in \mathrm{BG}_{z,t}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h)$$
$$= 2\left\lfloor \frac{t-z}{2} \right\rfloor (f_t(x^2q^{2t}))^{\lfloor \frac{t-z}{2} \rfloor - 1} g_t(x^2q^{2t}) \prod_{i=0}^{\lfloor (t-z)/2 \rfloor - 1} (-q^{2i+z+1}, -q^{2t-2i-z-1}, q^{2t}; q^{2t})_\infty$$

Hook lengths and congruences

$n_t(\lambda)$: number of hooks of length t in the partition λ

Remark: $n_1(\lambda)$ corresponds to the number of distinct antecedents of λ in the Young Lattice and $n_t(\lambda)$ is the number of paths of the form $s_i \cdots s_{i+t}$ ending at λ in the Young lattice

Theorem (Han–Ji, 2011): $\displaystyle\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} y^{n_1(\lambda)} = \frac{((1-y)q;q)_\infty}{(q;q)_\infty}$,

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$

$a_t^*(n)$: number of hooks of length t among the self-conjugate partitions of n. When t is even, $\mathcal{P}_0 = \mathrm{BG}_{0,t}$

Theorem (Amdeberhan–Andrews–Ono–Singh, 2024): $\forall n \geq 0$, $a_{2t}^*(n) \equiv 0 \pmod{2t}$.

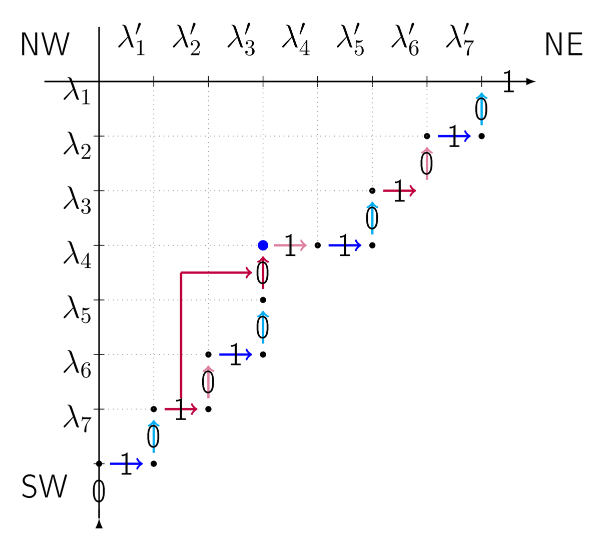
A key tool: the Littlewood decomposition

Set $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{A}_{(t)} := \{\omega_t \in \mathcal{A} \mid \mathcal{H}_t(\omega_t) = \emptyset\}$
Littlewood decomposition: bijection such that

$$\lambda \in \mathcal{P} \mapsto (\omega_t, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$$
$$\mathcal{H}_t(\lambda) = t \bigcup_{i=0}^{t-1} \mathcal{H}(\nu^{(i)})$$
$$|\lambda| = |\omega_t| + t \sum_{i=0}^{t-1} |\nu^{(i)}|$$

Restriction to $\mathrm{BG}_{z,t}$
$\lambda \in \mathcal{P} \mapsto (\omega_t, \underline{\nu}) \in \mathrm{BG}_{z,t,(t)} \times \mathcal{P}^{\lfloor (t-z)/2 \rfloor}$

Remark: Albion(2026) when $|z| > 1$, the t quotient of a z-asymmetric partition heavily depends on the core. This implies that we do not have theorems as nice as above on the set of z-asymmetric partitions.



A corollary when $\rho_1 = 1$

$$\sum_{\lambda \in \mathrm{BG}_{z,t}} q^{|\lambda|} y^{n_t(\lambda)} = ((1-y^2)q^{2t};q^{2t})_\infty^{\lfloor \frac{t-z}{2} \rfloor} \prod_{i=0}^{\lfloor (t-z)/2 \rfloor - 1} (-q^{2i+z+1}, -q^{2t-2i-z-1}; q^{2t})_\infty$$

Applications

$a_{z,t}(n)$: number of hooks of length t among all partitions of n in $\mathrm{BG}_{z,t}$

$$a_{z,t}(n) = 2\lfloor (t-z)/2 \rfloor \sum_{j \geq 1} |\mathrm{BG}_{z,t}(n - 2tj)| \quad \text{and} \quad a_{z,t}(n) \equiv 0 \pmod{2\lfloor (t-z)/2 \rfloor}$$

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