Lattice Path Conference
20-24 July 2026
TU Wien
Vienna, Austria

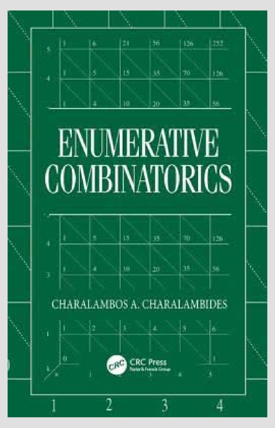
Charalambos A. Charalambides (1945–2024), q-distributions, and advances in q-order statistics

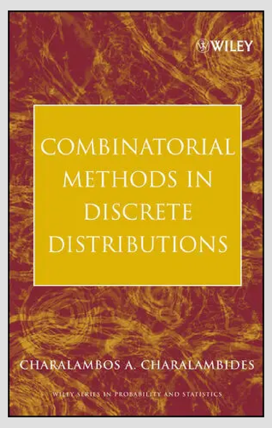
Malvina Vamvakari
Dept. Informatics & Telematics, Harokopio University of Athens, Greece

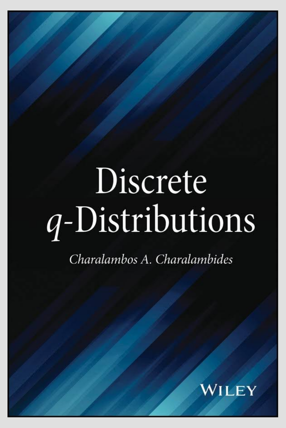
Charalambos A. Charalambides (1945–2024)

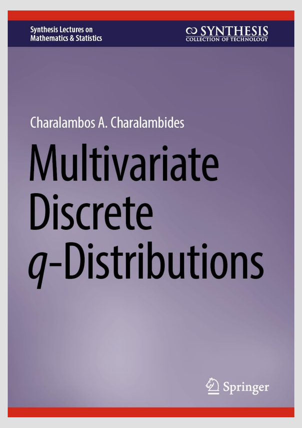
- Born: 19 December 1945, Koilani, Limassol, Cyprus.
- B.Sc. in Mathematics, National and Kapodistrian University of Athens (1969).
- Ph.D. in Statistics (1972), supervised by Professor Th. Kakoullos.
- Joined the Department of Mathematics in 1971; full professor in 1987.
- Visiting professor at McGill University, Temple University, and the University of Cyprus.
- Research interests: combinatorics, discrete distributions, parameter estimation, probability and statistics.
- Author of four research monographs, several textbooks, and ~ 70 journal articles.
- His scientific legacy continues to inspire researchers and students.

Research monographs in enumerative combinatorics and q-distributions

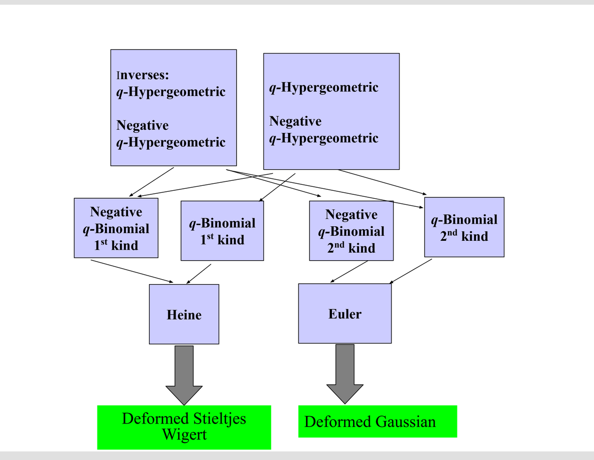




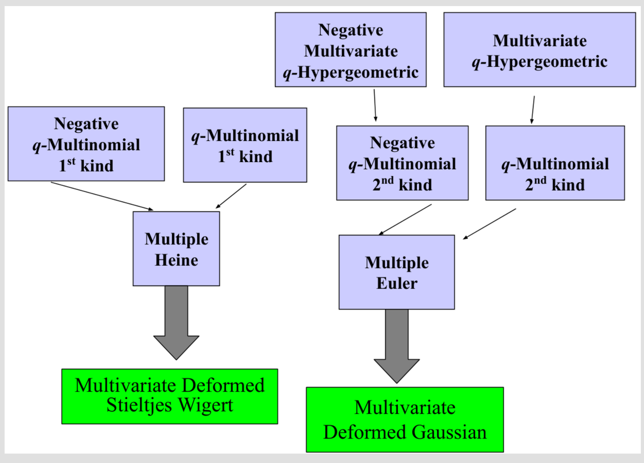




Univariate and multivariate discrete q-distributions (Charalambides, 2016, 2024)



Univariate discrete q-distributions:
stochastic models of sequences of n independent Bernoulli trials with success probability varying geometrically (with rate q) either with the number of previous trials or with the number of previous successes, or both.



Multivariate discrete q-distributions:
stochastic models of sequences of n independent Bernoulli trials with chain-composite successes, where the odds of success of a certain kind at a trial is assumed to vary geometrically (with rate q) with the number of previous trials or with the number of previous successes, or both.

q-Multinomial distribution of the 1st kind

- X_j: #successes of a jth kind in a sequence of n independent Bernoulli trials with chain composite failures, where the probability of success of the jth kind at the ith trial is given by
$$p_{j,i} = \frac{\theta_j q^{i-1}}{1+\theta_j q^{i-1}},\ 0<\theta_j<\infty,\ j\geq 1, i\geq 1,\ 0<q<1.$$
- Joint probability function of the random vector $\mathcal{X}=(X_1,\ldots,X_k)$ is given by
$$f^B_{\mathcal{X}}(x_1,\ldots,x_k)=P(X_1=x_1,\ldots,X_k=x_k)=\binom{n}{x_1,\ldots,x_k}_q \prod_{j=1}^k \frac{\theta_j^{x_j} q^{\binom{x_j}{2}}}{\prod_{i=1}^{n-s_{j-1}}(1+\theta_j q^{i-1})}$$
with $\sum_{j=1}^k x_j \leq n$ and $s_j=\sum_{i=1}^j x_i$.

The discrete limit of the joint p.f. of the q-multinomial distribution of the 1st kind is the multiple Heine distribution:
$$\lim_{n\to\infty}\binom{n}{x_1,\ldots,x_k}_q \prod_{j=1}^k \frac{\theta_j^{x_j} q^{\binom{x_j}{2}}}{\prod_{i=1}^{n-s_{j-1}}(1+\theta_j q^{i-1})}=\prod_{j=1}^k \frac{q^{\binom{x_j}{2}}\lambda_j^{x_j}}{[x_j]_q!}\prod_{i=1}^{\infty}(1+\lambda_j(1-q)q^{i-1})^{-1},\ x_j=0,\ldots,\ \lambda_j>0,\ \lambda_j=\theta_j/(1-q).$$

q-Multinomial distribution of the 2nd kind

- Y_j: number of successes of a jth kind in a sequence of n independent Bernoulli trials with chain composite failures, where the conditional probability of success of the jth kind at any trial, given that $i-1$ successes of the jth kind occur in the previous trials is given by
$$p_{j,i}=1-\theta_j q^{i-1}, j\geq 1, i\geq 1, 0<\theta_j<1.$$
- Joint probability function of the random vector $\mathcal{Y}=(Y_1,\ldots,Y_k)$:
$$f^{MS}_{\mathcal{Y}}(y_1,\ldots,y_k)=P(Y_1=y_1,\ldots,Y_k=y_k)=\binom{n}{y_1,\ldots,y_k}_q \prod_{j=1}^k \theta_j^{y_j}\prod_{i=1}^{n-s_j}\left(1-\theta_j q^{i-1}\right),$$

$y_j=0,1,\ldots,n,\ \sum_{j=1}^k y_j\leq n,\ s_j=\sum_{i=1}^j y_i$.

The discrete limit of the joint p.f. of the q-multinomial distribution of the 2nd kind is the multiple Euler distribution:
$$\lim_{n\to\infty}\binom{n}{y_1,\ldots,y_k}_q \prod_{j=1}^k \theta_j^{y_j}\prod_{i=1}^{n-s_j}\left(1-\theta_j q^{i-1}\right)=\prod_{j=1}^{\infty}\left(1-\lambda_j(1-q)q^{i-1}\right)\frac{\lambda_j^{y_j}}{[y_j]_q!},\ \lambda_j=\theta_j/(1-q).$$

Multivariate q-hypergeometric distribution

- Consider an urn containing ν balls, $\{b_1,\ldots,b_\nu\}$, of $k+1$ different ordered colors, with ν_ℓ distinct balls of color c_ℓ, $\{b_{s_{\ell-1}+1}, b_{s_{\ell-1}+2},\ldots,b_{s_\ell}\}$, for $\ell=1,\ldots,k+1$, where $s_0=0$, and
$$s_\ell=\sum_{i=1}^{\ell}\nu_i,\quad \text{for } \ell=1,\ldots,k+1,$$
with $s_{k+1}=\nu$. The first k colors, $\{c_1,\ldots,c_k\}$, may be considered as shades of white, and color $\{c_{k+1}\}$ as black.
- W_ℓ : number of balls of color c_ℓ drawn in n q-drawings in a multiple q-hypergeometric urn model, with the conditional probability of drawing a ball of color c_ℓ at the ith q-drawing, given that $j_\ell-1$ white balls of color c_ℓ and a total of colors $c_1,\ldots,c_{\ell-1}$ are drawn in the previous $i-1$ q-drawings:
$$p_{i,j_\ell}(\ell)=\frac{[\nu_\ell-j_\ell+1]_q}{[\nu+s_\ell-i+1]_q}, j_\ell=1,\ldots,i,\ell=1,\ldots,k,i\geq 1.$$
- Joint probability function of a random vector $\mathcal{W}=(W_1,\ldots,W_k)$:
$$f_{\mathcal{W}}(w_1,\ldots,w_k)=P(W_1=w_1,\ldots,W_k=w_k)=\binom{n}{w_1,\ldots,w_k}_q q^{\sum_{j=1}^k(n-\sum_{i=1}^j w_i)(\nu_j-w_j)}\frac{\prod_{j=1}^{k+1}[\nu_j]_{w_j,q}}{[\nu]_{n,q}}$$

$w_j=0,\ldots,n,\ j=0,\ldots,k$, with $\sum_{j=1}^k n_j\leq n$, where $w_k=n-\sum_{j=1}^k w_j, \nu=\sum_{j=1}^{k+1}\nu_j$, and $n\leq[\nu_j]$, $j=1,\ldots,k$.

On q-continuous random variables, K & V' 17, V'23

A k-variate random variable $\mathcal{X}=(X_{1,q},\ldots,X_{k,q})$ is called q-continuous if there exists a non-negative function
$$f_q(x_1,\ldots,x_k)\geq 0,\ x_i>0,\ i=1,\ldots k$$
with
$$\int_0^\infty\cdots\int_0^\infty\int_0^\infty f_q(x_1,\ldots,x_k)d_q x_1\cdots d_q x_k=1,$$
where for every real numbers α_i, β_i with $\alpha_i<\beta_i$, $i=1,\ldots,k$
$$P\left(\alpha_1<X_{1,q}\leq\beta_1,\ldots,\alpha_k<X_{k,q}\leq\beta_k\right)=\int_{\alpha_k}^{\beta_k}\cdots\int_{\alpha_1}^{\beta_1}f_q(x_1,\ldots,x_k)d_q x_1\cdots d_q x_k.$$
The function $f_q(x_1,\ldots,x_k)\geq 0$ is called q-density function of the k-variate random variable $\mathcal{X}=(X_{1,q},\ldots,X_{k,q})$ or **joint q-density function** of the random variables $X_{1,q},\ldots,X_{k,q}$.
For the corresponding joint distribution function, we have:
$$F(x_1,\ldots,x_k)=P\left(X_{1,q}\leq x_1,\ldots,X_{k,q}\leq x_k\right)=\int_0^{x_k}\cdots\int_0^{x_1}f_q(t_1,\ldots,t_k)d_q t_1\cdots d_q t_k.$$
Taking the q-derivative of the above relation we have
$$\frac{\partial F(x_1,\ldots,x_k)}{\partial_q x_k\cdots\partial_q x_1}=f_q(x_1,\ldots,x_k)=\frac{P\left(qx_1<X_{1,q}\leq x_1,\ldots,qx_k<X_{k,q}\leq x_k\right)}{(1-q)x_1\cdots(1-q)x_k}.$$

On Heine stochastic process, K & V' 17

- Consider a stochastic model that is developing in time or space, and let $X(t), t\geq 0$, be the number of successes (occurrence of event A) in the interval $(0,t]$. Assume that $X(t), t\geq 0$, is a stochastic process, with independent and homogeneous increments, which starts at time $t=0$ from state 0, $P\left(X(0)=0\right)=1$, and, in the small time interval $(qt,t]$, of length $\delta t=(1-q)t$, satisfies the condition
$$p_j(\delta t)=P\left(X(t)-X(qt)=j\right)=\begin{cases}\frac{1}{1+\lambda(1-q)t}, & j=0,\\ \frac{\lambda(1-q)t}{1+\lambda(1-q)t}, & j=1,\\ 0, & j>1,\end{cases}$$
with $0<\lambda<\infty$. Then, $X(t), t\geq 0$, is called Heine process, with parameters λ and q.
- The probability function of the Heine process, $X(t), t\geq 0$, with parameters λ and q, is given by
$$P(X(t)=k)=e_q(-\lambda t)\frac{q^{\binom{k}{2}}(\lambda t)^k}{[k]_q!},$$
for $k\in\mathbb{N}$, where $e_q(z)=\prod_{i=1}^{\infty}\left(1-(1-q)zq^{i-1}\right)^{-1}$, $|z|<1/(1-q)$ is the small q-exponential function.
- The distribution function of the waiting times W_n, $n\geq 1$, in a Heine process is that of the q-Erlang distribution of the first kind, with parameters n,λ and q, given by
$$F_n(w)=P(W_n\leq w)=1-\sum_{x=0}^{n-1}e_q(-\lambda w)\frac{q^{\binom{x}{2}}(\lambda w)^x}{[x]_q!},0<w<\infty,$$
where n is a positive integer, $0<\lambda<\infty$. Its q-density function $f_n(w)=d_qF_n(w)/d_qw$ is given by
$$f_n(w)=\frac{q^{\binom{n}{2}}\lambda^n}{[n-1]_q!}t^{n-1}e_q(-\lambda w),0<w<\infty.$$
- The q-density function of the interarrival times $T_1=W_1$, $T_k=W_k-W_{k-1}$, $k\geq 2$, in a Heine process is that of the q-exponential distribution of the first kind, given by
$$f_{T_k}(t)=\lambda e_q(-\lambda t),\qquad \text{for } 0<t<\infty \text{ and } 0<\lambda<\infty.$$
- Let $\mathcal{W}=(W_{1,q},\ldots,W_{\nu,q})$, be a νth-variate random variable and $W_{k,q}$, $k=1,\ldots\nu$ be the random variables of the waiting times in a Heine process. Then the joint q-density function of the random variables $W_{k,q}$, $k=1,\ldots\nu$, is given by
$$f_{\mathcal{W}}(w_1,\ldots,w_\nu)=\lambda^\nu\prod_{j=1}^{\nu}e_q\left(-\lambda(w_j-w_{j-1})\right),w_0=0,$$
$0<w_1<\ldots<w_\nu<\infty$, where $0<\lambda<\infty$.

Motivation: a theorem leading to q-order statistics, V'23

Let W_n be the waiting time of the nth arrival of the Heine process $\{X(t),\ t>0\}$ with parameters λ and q. Then the joint q-density function of the waiting times $W_1,\ldots,W_\nu$, in which the first ν events occur given that $X(t)=\nu$, $0<w_1<\cdots<w_\nu<t$ with $w_i\in(q^{\nu-i+1}t,q^{\nu-i}t]$, $i=1,\ldots,\nu-1$, is given by
$$f_q\left(w_1,\ldots,w_\nu|X(t)=\nu\right)=\frac{[\nu]_q!}{q^{\binom{\nu}{2}}t^\nu}.$$

Classical order statistics

Let a ν-variate random variable $\mathcal{X}=(X_1,\ldots,X_\nu)$ be defined in a sample space Ω. Then for the values $x_1=X_1(\omega),\ldots,x_\nu=X_\nu(\omega)$, $\omega\in\Omega$ there is a permutation $(i_1,\ldots,i_\nu)$ of the ν indices $\{1,\ldots,\nu\}$, such that $x_{i_1}\leq\cdots<x_{i_{\nu-1}}\leq x_{i_\nu}$. The k-th ordered random variable is denoted by $X_{(k)}$ and defined by $X_{(k)}(\omega)=x_{(k)}$, $\omega\in\Omega$, where $x_{(k)}=x_{i_k}$, $k=1,\ldots,\nu$. In particular, for $k=1$ this gives $X_{(1)}=\min\{X_1,\cdots,X_\nu\}$ and, for $k=\nu$ this gives $X_{(\nu)}=\max\{X_1,\ldots,X_\nu\}$. Generally, $X_{(1)}\leq\cdots\leq X_{(\nu)}$.

q-Order statistics (V'23), dependencies & patterns (V'25)

Let $\mathcal{Y}=(Y_1,\ldots,Y_\nu)$ be a ν-variate random variable and $Y_{(k)}$, be the corresponding k-th ordered random variables. Assume
$$0\leq Y_{(1)}<qY_{(2)}<Y_{(2)}<\cdots<Y_{(\nu-1)}<qY_{(\nu)}<Y_{(\nu)}\leq\beta,$$
then, $Y_{(k)}$ is called the k-th *q-ordered* random variables.

For any integer r between 1 and ν, let $\{i_1,\ldots,i_r\}$ be an r-combination of $\{1,\ldots,\nu\}$ satisfying $i_1<\cdots<i_r$, and let $\{i_{r+1},\ldots,i_\nu\}$ be its complementary combination (i.e., one has $\{i_1,\ldots,i_r\}\cup\{i_{r+1},\ldots,i_\nu\}=\{1,\ldots,\nu\}$) satisfying $i_{r+1}<\cdots<i_\nu$. Then, the non-ordered random variables Y_i's satisfy the following dependence relations for $y\in[0,\beta]$:
$$P\left(Y_{i_r}\leq y|Y_{i_1}\leq y,\ldots,Y_{i_{r-1}}\leq y\right)=P\left(Y_{i_r}\leq q^{\mathrm{des}(\mathcal{Y}_r)}y\right),\ \text{where}\ \mathrm{des}(\mathcal{Y}_r)=\sum_{j=1}^{r-1}\mathbf{1}_{Y_{i_j}>Y_{i_{j+1}}}\text{with}\ \mathcal{Y}_r=(Y_{i_1},\ldots,Y_{i_r}),$$
$$P\left(Y_{i_r}\leq y|Y_{i_1}>y,\ldots,Y_{i_{r-1}}>y\right)=P\left(Y_{i_r}\leq y\right),\qquad\text{and}$$
$$P\left(Y_{i_m}\leq y|Y_{i_1}\leq y,\ldots,Y_{i_r}\leq y,Y_{i_{r+1}}>y,\ldots,Y_{i_{m-1}}>y\right)=P\left(Y_{i_m,q}\leq q^{\mathrm{des}(\mathcal{Y}_m)}y\right),\ \text{with}\ \mathcal{Y}_m=\left(Y_{i_1},\ldots,Y_{i_m}\right),m=r+1,\ldots,\nu.$$

Results

- Distributional properties of q-order statistics.
- Joint and marginal q-densities for dependent random variables.
- Results for extrema, spacings, ratios, differences, and sample medians.
- Distributional properties for dependent, non-identically distributed q-uniform and discrete q-uniform random variables.
- Permutation statistics and their q-analogues as fundamental tools in the analysis.
- Extensions to q-Dirichlet, q-Beta, and Heine-process models.

References

Charalambides, C. (2016). *Discrete q-Distributions.* John Wiley & Sons.
Charalambides, C. (2024). *Multivariate Discrete q-Distributions.* Springer.
Kyriakoussis, A. and Vamvakari, M.G. (2017). *Heine process as a q-analog of the Poisson process: waiting and interarrival times.* Communication in Statistics Theory and Methods, 46(8), 4088-4102.
Vamvakari, M.G. (2023). *On q-Order Statistics.* Séminaire Lotharingien de Combinatoire, 87B (9),1–25. Lattice Path Combinatorics and Applications.
Vamvakari, M.G. (2025).*On Patterns in q-Order Statistics.* Permutation Patterns 2025, (University of St Andrews, Scotland, July 7–11), pp. 100–103.