Steady State and Absorbing Probabilities of Common Markov Chains

Heba Ayeda[1], Theodore De Santos[1], Sean Kanne[1], Alan Krinik[1], Gerardo Rubino[2], José M. Martínez[3]

[1]California State Polytechnic University, Pomona, USA, [2]INRIA, France, [3]UTFSM, Chile

1. Cool Alternative Algorithm to Find Unique Stationary Distributions

Let us describe this result for a 4-state irreducible one-step transition probability matrix P. Assume $\vec{\pi} = [\pi(0)\ \pi(1)\ \pi(2)\ \pi(3)]$ is its unique stationary distribution and suppose $\widetilde{P}$ is as defined below:

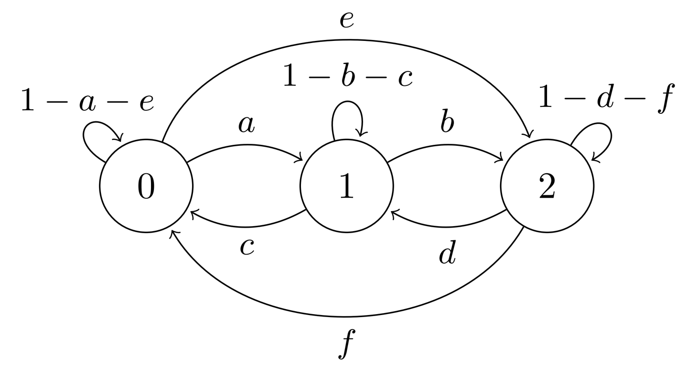
$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix}, \quad \widetilde{P} = \begin{bmatrix} P_{00}-1 & P_{01} & P_{02} & 1 \\ P_{10} & P_{11}-1 & P_{12} & 1 \\ P_{20} & P_{21} & P_{22}-1 & 1 \\ P_{30} & P_{31} & P_{32} & 1 \end{bmatrix}$$

Then, $\widetilde{P}$ is invertible and $\vec{\pi} = [0\ 0\ 0\ 1]\,\widetilde{P}^{-1}$. Moreover,

$$\pi(0) = \frac{N_0}{D}, \quad \pi(1) = \frac{N_1}{D}, \quad \pi(2) = \frac{N_2}{D}, \quad \pi(3) = \frac{N_3}{D},$$

where $N_i = \widetilde{M}_{i3}$, $\widetilde{M}_{ij}$ is the minor of the ij entry of $\widetilde{P}$, and D is the determinant of $\widetilde{P}$. The result is due to Mihoc (1934) and can be found in [5]; special thanks to Stewart Ethier for this reference.

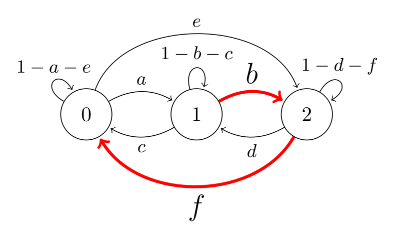
2. Steady-State Distribution of General 3-state Markov Chain



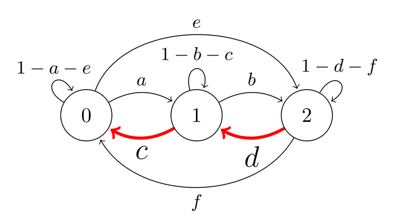
$$\begin{bmatrix} \pi(0) \\ \pi(1) \\ \pi(2) \end{bmatrix} = \begin{bmatrix} \frac{bf+cd+cf}{ab+ad+af+be+bf+ce+cd+cf+de} \\ \frac{ad+af+de}{ab+ad+af+be+bf+ce+cd+cf+de} \\ \frac{ab+be+ce}{ab+ad+af+be+bf+ce+cd+cf+de} \end{bmatrix}$$

3. Steady-State and Oriented Spanning Trees

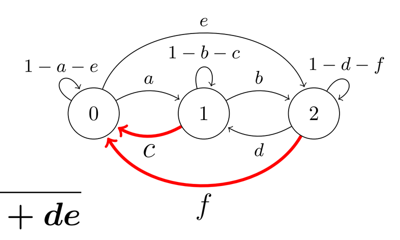
As a Markov chain is a special case of a weighted digraph and those we consider in this talk are strongly connected, we can construct their stationary distribution using the Markov Chain Tree Theorem [4] via:



$$\pi(i) = \frac{t_i}{\sum_{j=0}^{n-1} t_j}, \quad t_i = \sum_{T \in \mathbb{T}_i} \prod_{e \in E(T)} P_e,$$



where $\mathbb{T}_i$ is the set of oriented spanning trees rooted at i, $E(T)$ is the edge set of T, and P_e is the probability of edge e. Using $\pi(0)$ from the general 3-state chain as an example:



$$\pi(0) = \frac{bf+cd+cf}{ab+ad+af+be+bf+ce+cd+cf+de}$$

4. Siegmund dual (Exponential-Dual in [6])

Assume $P = [P_{i,j}]$ for $i, j \in \{0, 1, \ldots, H\}$ is a $(H+1) \times (H+1)$ stochastic matrix. Suppose that the $(H+2) \times (H+2)$ matrix $P^* = [P^*_{i,j}]$ defined by

$$P^*_{i,j} = \sum_{s=i+1}^{H} [P_{j+1,s} - P_{j,s}]$$

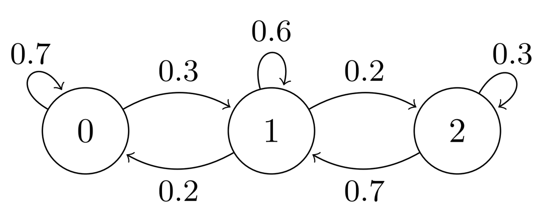
is also a stochastic matrix, with the following conventions:

- $P_{-1,s} = 0$ if $-1 < s \le H$,
- $P^*_{H,H} = 1$,
- $P^*_{i,H} = 1 - \sum_{s=0}^{H-1} P^*_{i,s}$.

Then P^* is called the dual matrix of P. Note that P^* is not necessarily a stochastic matrix. P and P^* have the same spectra.

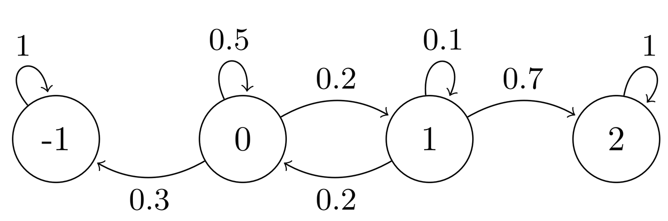
Suppose,

$$P = \begin{array}{c} 0\quad 0\quad 0\quad 0 \\ \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0.7 & 0.3 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0\quad 0\quad 0\quad 1 \end{array}$$



Then,

$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 \\ 0 & 0.2 & 0.1 & 0.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



5. Using duality

After evaluating P^*, we can use the Duality Theorem to calculate each element of the given P matrix, and of its powers, and viceversa:

$$P^{*(k)}_{i,j} = \sum_{s=i+1}^{H} \left[P^{(k)}_{j+1,s} - P^{(k)}_{j,s}\right] \qquad P^{(k)}_{i,j} = \sum_{s=i}^{H} \left[P^{*(k)}_{j,s} - P^{*(k)}_{j-1,s}\right]$$

for $n \ge 0$ and for all states $i, j = 0, 1, 2, \ldots, H$ with the conventions:

$$P^{*(k)}_{-1,-1} = 1 \qquad P^{*(k)}_{H,H} = 1$$

In particular, the finite time Gambler's Ruin probability is:

$$P^{*(k)}_{i,-1} = \sum_{s=i+1}^{H} P^{(k)}_{0,s} \qquad \text{where } i = 0,\ 1,\ 2,\ \cdots,\ H-1.$$

Assume P is the 1-step transition probability matrix of an irreducible finite Markov chain. We know that there exists a steady state distribution $\vec{\pi} = [\pi(0), \pi(1), \pi(2), \ldots, \pi(H)]$. Then,

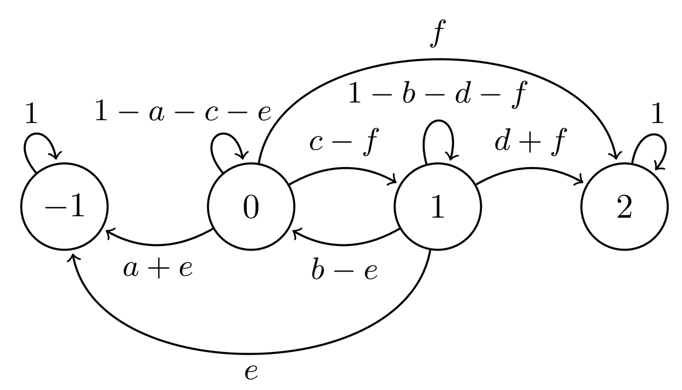
$$\lim_{k\to\infty} \sum_{s=i+1}^{H} P^{(k)}_{0,s} = \sum_{s=i+1}^{H} \lim_{k\to\infty} P^{(k)}_{0,s} = \sum_{s=i+1}^{H} P^{(\infty)}_{0,s}$$
$$= \sum_{s=i+1}^{H} \pi(s) = \lim_{k\to\infty} P^{*(k)}_{i,-1} = P^{*(\infty)}_{i,-1}$$

where $P^{*(\infty)}_{i,-1}$ is the Gambler's Ruin probability of, starting at i in the dual, getting absorbed at -1. So,

$$P^{*(\infty)}_{i,-1} = \sum_{s=i+1}^{H} \pi(s).$$

6. Absorbing probabilities in the 3-state case

The dual of the general chain with 3 states is



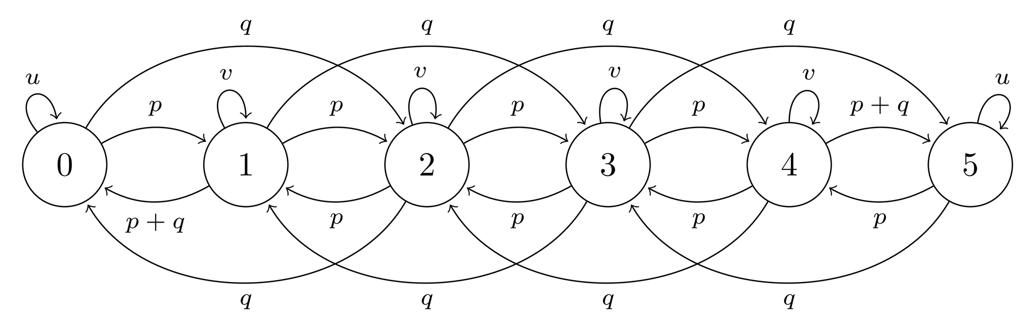
We have:

$$P^{*\infty}_{0,-1} = \pi(1) + \pi(2) = \frac{ad+af+de+ab+be+ce}{ab+ad+af+be+bf+ce+cd+cf+de}$$

$$P^{*\infty}_{1,-1} = \pi(2) = \frac{ab+be+ce}{ab+ad+af+be+bf+ce+cd+cf+de}$$

7. A 6-state family

More patterns: consider



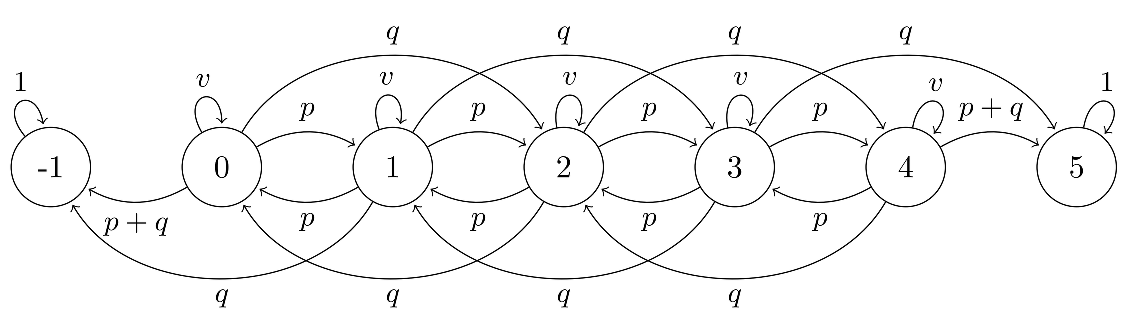
$$u = 1 - p - q \qquad v = 1 - 2p - 2q$$

$$\pi(0) = \pi(5) = \frac{p^2 + 4pq + 3q^2}{6p^2 + 20pq + 12q^2}$$

$$\pi(1) = \pi(4) = \frac{p^2 + 3pq + q^2}{6p^2 + 20pq + 12q^2}$$

$$\pi(2) = \pi(3) = \frac{p^2 + 3pq + 2q^2}{6p^2 + 20pq + 12q^2}$$

Dual:



Due to symmetry, we have:

$$P^{*\infty}_{0,-1} = 1 - P^{*\infty}_{4,-1} = \sum_{i=1}^{5} \pi(i) = \frac{5p^2 + 16pq + 9q^2}{6p^2 + 20pq + 12q^2}$$

$$P^{*\infty}_{1,-1} = 1 - P^{*\infty}_{3,-1} = \sum_{i=2}^{5} \pi(i) = \frac{4p^2 + 13pq + 8q^2}{6p^2 + 20pq + 12q^2}$$

$$P^{*\infty}_{2,-1} = \sum_{i=3}^{5} \pi(i) = \frac{1}{2}$$

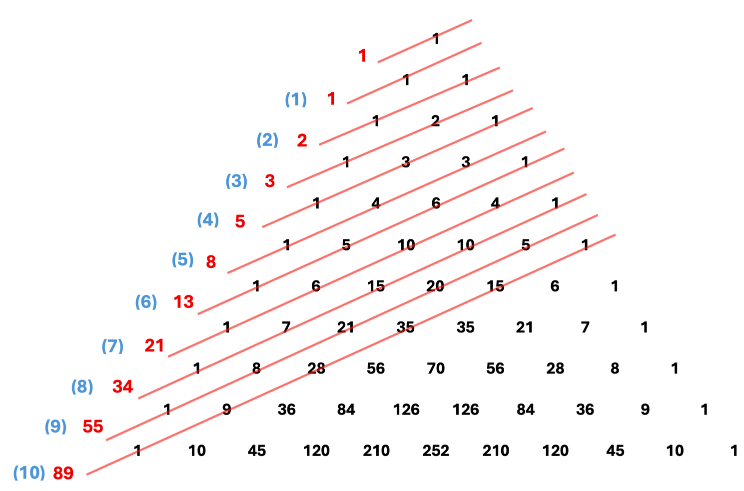
8. Same topology but with n states

Consider a chain with the same characteristics as in previous section, but with an arbitrary number n of states.

Let $\pi(i) = N_{i,n}/D_n$ be the steady state of the ith node of the chain with n nodes **in simplified form**; for instance,

$$N_{0,9} = N_{8,9} = p^4 + 7p^3q + 15p^2q^2 + 10pq^3 + q^4$$
$$N_{1,9} = N_{7,9} = p^4 + 6p^3q + 10p^2q^2 + 4pq^3$$

The coefficients of the end states of these Markov chains follow "up 1, over 1" steps along Pascal's Triangle.



These also describe the Fibonacci Polynomials, which are defined by

$$F_1(x) = 1, \quad F_2(x) = x, \quad F_n(x) = xF_{n-1}(x) + F_{n-2}(x).$$

9. Fibonacci

The numerators of the simplified steady state for the two end nodes satisfy (for $i \in \{0, 1\}$):

$$N_{i,n} = N_{n-i-1,n} = \sum_{e=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-e-i-1}{e} p^{n-e} q^{e}.$$

More generally, the coefficients of these numerators can be represented using the coefficients of the products of Fibonacci polynomials. These allow us to define the coefficients of the simplified steady state of node i in a symmetric chain of (odd) length n as the coefficients of

$$\left(F_{\frac{n+1}{2}}(x)\right)^2 + (-1)^i \cdot \left(F_{\frac{n-1}{2}-i}(x)\right)^2$$

while defining the coefficients of node i in a symmetric chain of (even) length n as the coefficients of

$$F_{\frac{n}{2}}(x) \cdot F_{\frac{n}{2}+1}(x) + (-1)^i \cdot F_{\frac{n}{2}-i-1}(x) \cdot F_{\frac{n}{2}-i}(x).$$

10. Conclusions

- Duality also can help in solving Gambler Ruin's problems.
- Moreover, regularities (patterns) appear when considering specific topologies (queuing models with bulk arrivals and more symmetries).
- All this is work in progress.

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