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Affine weighted Motzkin paths and the differential kernel method

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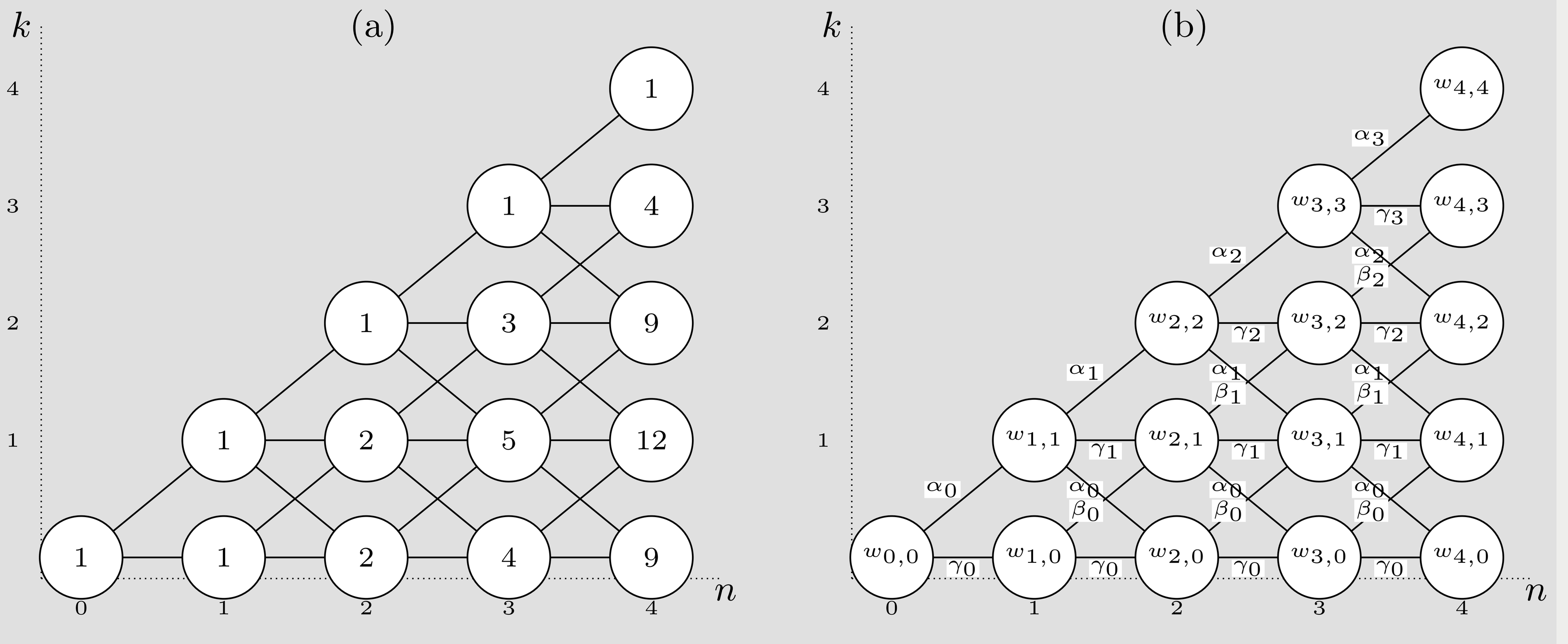
1. The model and the return problem

A **Motzkin path** starts at height 0 and stays nonnegative. An up-step leaving height k has weight α_k, a down-step landing at height k has weight β_k, and a level-step at height k has weight γ_k.



The classical Motzkin triangle and a height-dependent weighted triangle.

Let $w_{n,k}$ be the total weight of length-n paths ending at height k, and set

$$r_n := w_{n,0}, \qquad W_k(t) := \sum_{n\ge 0} w_{n,k}\frac{t^n}{n!}, \qquad W(x,t) := \sum_{k\ge 0} W_k(t)x^k,$$

$$W_0(t) := W(0,t) = \sum_{n\ge 0} r_n\frac{t^n}{n!}.$$

We use **affine weights**

$$\alpha_k := ak+\alpha_0, \qquad \beta_k := bk+\beta_0, \qquad \gamma_k := ck+\gamma_0, \qquad Q(x) := ax^2+cx+b.$$

The path recurrence becomes

$$\boxed{\partial_t W(x,t) = Q(x)\partial_x W(x,t) + (\alpha_0 x+\gamma_0)W(x,t) + (\beta_0-b)\frac{W(x,t)-W_0(t)}{x}.}$$

$b=\beta_0$: classical PDE **$b<\beta_0$: one more unknown function $W_0(t)$.**

2. From the PDE to one equation for $W_0(t)$

Assume $0<b<\beta_0$ and set

$$\nu := \frac{\beta_0-b}{b} > 0.$$

Fix $T>0$ and consider W along the curve

$$\tau \longmapsto (X(\tau), T-\tau), \qquad X(0) := 0,$$

by setting

$$U_T(\tau) := W(X(\tau), T-\tau).$$

The chain rule and the PDE give

$$U_T'(\tau) = \big(X'(\tau) - Q(X(\tau))\big)W_x(X(\tau), T-\tau)$$
$$-(\alpha_0 X(\tau)+\gamma_0)U_T(\tau) - \frac{\beta_0-b}{X(\tau)}\big(U_T(\tau) - W_0(T-\tau)\big).$$

Choose

$$X'(\tau) := Q(X(\tau)), \qquad X(0) := 0.$$

The red term disappears, and the PDE becomes the linear ODE

$$U_T'(\tau) + \left(\alpha_0 X(\tau)+\gamma_0+\frac{\beta_0-b}{X(\tau)}\right)U_T(\tau) = \frac{\beta_0-b}{X(\tau)}W_0(T-\tau).$$

Treating W_0 temporarily as known, we solve this inhomogeneous ODE by variation of constants. Define the explicit functions

$$H(\tau) := X(\tau)^\nu \exp\int_0^\tau \big[(\alpha_0-\nu a)X(r)+\gamma_0-\nu c\big]\,dr,$$

$$L(\tau) := \frac{\beta_0-b}{X(\tau)}H(\tau).$$

Variation of constants gives the solution that remains finite at the endpoint $\tau=0$:

$$\boxed{H(\tau)U_T(\tau) = \int_0^\tau L(v)W_0(T-v)\,dv.}$$

At $\tau=T$,

$$U_T(T) = W(X(T),0) = 1$$

by the initial condition $W(x,0)=1$. Hence the catalytic variable has disappeared:

$$\boxed{H(T) = \int_0^T L(v)W_0(T-v)\,dv.}$$

This one-variable equation computes the axis numbers successively. Write

$$H(T) = T^\nu\sum_{j\ge 0} h_j T^j, \qquad L(T) = T^{\nu-1}\sum_{j\ge 0}\ell_j T^j.$$

Then $r_0=1$, and for $N\ge 1$,

$$\boxed{r_N = \frac{N!}{\ell_0 B(\nu, N+1)}\left[h_N - \sum_{j=1}^{N}\ell_j\frac{r_{N-j}}{(N-j)!}B(\nu+j, N-j+1)\right],}$$

where B is the beta function.
After W_0 is known, set

$$\rho(x) := \int_0^x \frac{du}{Q(u)}.$$

The same identity, with $\tau=\rho(x)$ and $T=t+\rho(x)$, gives

$$\boxed{W(x,t) = \frac{1}{H(\rho(x))}\int_0^{\rho(x)} L(v)W_0(t+\rho(x)-v)\,dv.}$$

Classical kernel method: choose a root that cancels the unknown bivariate series. *Here:* choose $X'=Q(X)$ to cancel the derivative term; the finite endpoint condition then yields the one-variable equation for W_0.

Eliminate the catalytic variable $x \longrightarrow$ compute r_n and $W_0(t) \longrightarrow$ obtain $W(x,t)$.

3. Integer $\nu=m$: solve locally and erase the virtual levels

If $\nu=m\in\mathbb{Z}_{>0}$, set $j:=k+m$. If only $j\ge m$ is allowed, this merely shifts the original triangle upward:

$$x^m W(x,t) = \text{the original triangle on the shifted height scale.}$$

Extend the shifted weights to every $j\ge 0$:

$$\widetilde{\alpha}_j := aj+\alpha_0-ma, \qquad \widetilde{\beta}_j := b(j+1), \qquad \widetilde{\gamma}_j := cj+\gamma_0-mc.$$

Let $F(x,t)$ count this free shifted model. Unlike the equation for W, it contains no additional unknown function W_0:

$$\boxed{F_t = Q(x)F_x + \big((\alpha_0-ma)x+\gamma_0-mc\big)F, \qquad F(x,0)=x^m.}$$

Thus F is obtained from a **classical local PDE**.
The levels $j\ge m$ form the visible region; $0\le j<m$ are virtual. A virtual path can influence the visible region only by the jump

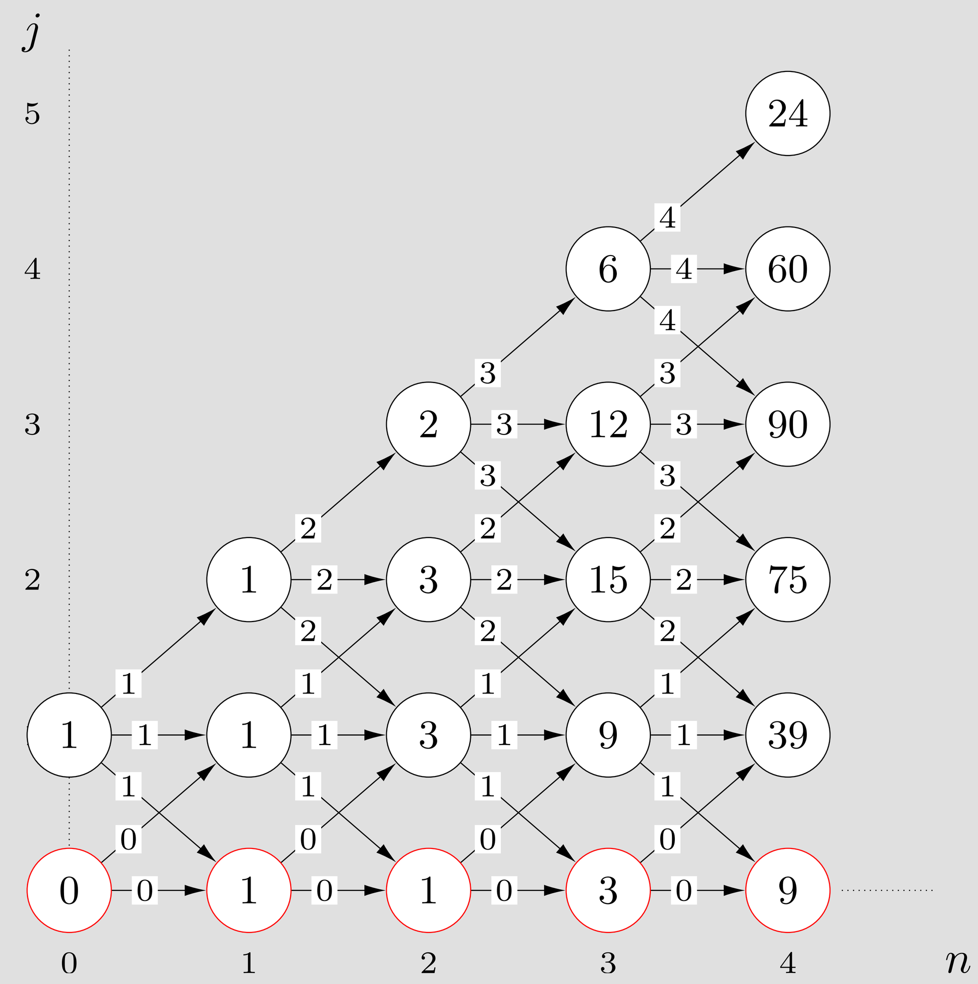
$$m-1 \longrightarrow m, \qquad \text{of weight } \alpha_0-a.$$

Define the visible part of F by

$$F_{\ge m}(x,t) := F(x,t) - \sum_{j=0}^{m-1}[x^j]F(x,t)\,x^j.$$

If $\alpha_0=a$, the jump $m-1\to m$ has weight 0, so virtual paths cannot return to a visible level. Therefore the desired generating function is obtained directly from the simpler function F:

$$\boxed{W(x,t) = x^{-m}F_{\ge m}(x,t) = x^{-m}\left(F(x,t) - \sum_{j=0}^{m-1}[x^j]F(x,t)x^j\right).}$$



Example $m=1$, $\alpha_0=a$: the red row is virtual and the jumps $0\to 1$ have weight 0.

If $\alpha_0\ne a$, virtual paths may return to visible levels; then their contributions must be erased after each row, rather than only at the end.

For $\alpha_0=a$: solve the local PDE for F, erase the powers $x^0,\dots,x^{m-1}$, and divide by x^m to obtain $W(x,t)$.

4. Factorisation and continued fractions

The same condition $\alpha_0=a$ has another consequence. In the Dyck example $\alpha_k=k+1$, $\beta_k=k+3$, $\gamma_k=0$,

$$W_k(t) = \sec^3 t\,\tan^k t.$$

Suppose more generally that

$$W_k(t) = W_0(t)Y(t)^k \qquad (k\ge 0).$$

The column equations at heights $0,1,2$ force

$$W_0' = (\gamma_0+\beta_0 Y)W_0,$$
$$Y' = \alpha_0+cY+bY^2,$$
$$(\alpha_0-a)W_0Y = 0.$$

Hence, in the nondegenerate case,

$$\boxed{W_k = W_0Y^k \text{ for all } k \iff \alpha_0=a.}$$

Then

$$Y' = a+cY+bY^2, \qquad W(x,t) = \frac{W_0(t)}{1-xY(t)}.$$

For $\alpha_k=k+1$, $\beta_k=k+p$, $\gamma_k=0$,

$$W_k(t) = \sec^p t\,\tan^k t.$$

The formal excursion ordinary generating function

$$E(z) := \sum_{n\ge 0} r_n z^n$$

has the Jacobi continued fraction

$$E(z) = \cfrac{1}{1-\gamma_0 z - \cfrac{\alpha_0\beta_0 z^2}{1-\gamma_1 z - \cfrac{\alpha_1\beta_1 z^2}{1-\gamma_2 z - \cdots}}}.$$

Thus excursions remember the level weights γ_k and the products $\alpha_k\beta_k$, but not the orientation of the two edge weights.

The condition $\alpha_0=a$ prevents the jump $m-1\to m$ and factorises all height columns.

Selected references

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