Off-diagonally symmetric alternating sign matrices

Nishu Kumari (University of Vienna)

Alternating sign matrices

- An alternating sign matrix (ASM) is a square matrix that satisfies the following conditions:
  - each entry is 1, 0 or -1,
  - the nonzero entries along each row and column alternate in sign,
  - the sum of the entries in each row and column is 1.
- Example:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- ASMs generalize permutation matrices.
- Introduced by Robbins and Rumsey in the 1980's while studying a certain generalization of the classical determinant.
- In mid-1980's, Mills, Robbins and Rumsey conjectured the following product formula for the number of ASMs.

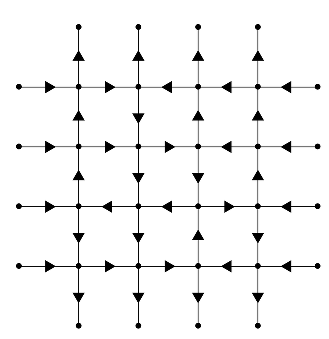
Theorem (Zeilberger, 1996)

The number of ASMs of order n is given by

$$|\mathrm{ASM}(n)| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

- Later, Kuperberg (1996) presented a completely different proof using methods from statistical physics.
- A six-vertex model configuration of size n (with domain wall boundary condition) is an $n \times n$ grid with arrows on each edge such that
  - at each bulk vertex (vertex of degree 4), there are as many incoming arrows as outgoing ones,
  - all arrows on the top and bottom boundaries are outgoing,
  - on the left and right boundaries all arrows are incoming.
- Bijection between ASMs and six-vertex model configurations on a square grid:

$$A_{i,j} = \begin{cases} 1 & \mathbf{c}_{i,j} = \;, \\ -1 & \mathbf{c}_{i,j} = \;, \\ 0 & \mathbf{c}_{i,j} = \;,\;,\;,\;, \end{cases} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longleftrightarrow$$



where $\mathbf{c}_{i,j}$ is the local configuration at (i,j).

Diagonally symmetric alternating sign matrices

- A diagonally symmetric ASM (DSASM) is an ASM that remains invariant under matrix transposition.
- Denote the set of DSASMs of order n by $\mathrm{DSASM}(n)$.
- Example:

$$\mathrm{DSASM}(1) = \{(1)\}, \qquad \mathrm{DSASM}(2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

$$\mathrm{DSASM}(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

- (Robbins, 1980)

n	1	2	3	4	5	6	7	8
$\lvert\mathrm{DSASM}(n)\rvert$	1	2	5	2^4	67	$2^4 \cdot 23$	$2 \cdot 5 \cdot 263$	$2^3 \cdot 11 \cdot 277$

Theorem (Behrend–Fischer–Koutschan, 2023)

The number of DSASMs of order n is given by

$$|\mathrm{DSASM}(n)| = \operatorname*{Pf}_{\chi_{\mathrm{odd}}(n) \le i < j \le n-1} \left((j-i) \sum_{k=0}^{i} \frac{3 - \delta_{k,0}}{i+j-2k} \binom{i+j-2k}{i-k} \right),$$

where $\chi_{\mathrm{odd}}(n)$ is 1 or 0 according to whether n is odd or even and δ is the Kronecker delta.

Even-order off-diagonally symmetric alternating sign matrices

- An even-order off-diagonally symmetric ASM (OSASM), as defined by Kuperberg (2002), is a DSASM of even order such that all its diagonal entries are zero.
- Denote the set of OSASMs of order n by $\mathrm{OSASM}(n)$.
- Example:

$$\mathrm{OSASM}(2) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \quad \mathrm{OSASM}(4) = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

n	1	2	3	4	5
$\lvert\mathrm{OSASM}(2n)\rvert$	1	3	$2 \cdot 13$	$2 \cdot 17 \cdot 19$	$3 \cdot 5 \cdot 7 \cdot 19 \cdot 23$

Theorem (Kuperberg, 2002)

The number of OSASMs of order $2n$ is given by

$$|\mathrm{OSASM}(2n)| = \prod_{i=1}^{n} \frac{(6i-2)!}{(2n+2i)!}.$$

- Observe that it is not possible for each diagonal entry of a DSASM of odd order to be zero.

Odd-order off-diagonally symmetric alternating sign matrices

- An odd-order off-diagonally symmetric ASM (OSASM), as defined by Behrend, Fischer and Koutschan (2023), is a DSASM of odd order with exactly one nonzero diagonal entry.
- Example:

$$\mathrm{OSASM}(1) = \{(1)\}, \qquad \mathrm{OSASM}(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

n	0	1	2	3	4	5
$\lvert\mathrm{OSASM}(2n+1)\rvert$	1	2^2	2^5	$2^7 \cdot 5$	$2^8 \cdot 7 \cdot 19$	$2^{11} \cdot 3 \cdot 5 \cdot 7 \cdot 23$

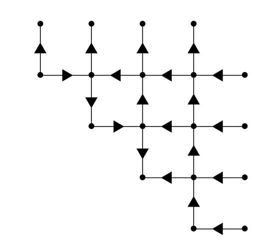
Theorem (Kumari, 2026)

The number of OSASMs of order $2n+1$ is given by

$$|\mathrm{OSASM}(2n+1)| = \frac{2^{n-1}(3n+2)!}{(2n+1)!} \prod_{i=1}^{n} \frac{(6i-2)!}{(2n+2i+1)!}.$$

- Sketch of the proof. Bijection between DSASMs and six-vertex model configurations on a triangular grid:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow$$



- Denote the set of six-vertex configurations on the triangular grid of size n by $6\mathrm{V}(n)$.
- Consider a partition function defined by

$$Z_n(u_1, \ldots, u_n) = \sum_{\mathbf{c} \in 6\mathrm{V}(n)} \left(\prod_{i=1}^{n} \mathrm{W}(\mathbf{c}_{i,i}, u_i) \prod_{1 \le i < j \le n} \mathrm{W}(\mathbf{c}_{i,j}, u_i u_j) \right),$$

where

$$\mathrm{W}(\;, u_i) = \mathrm{W}(\;, u_i) = s, \; \mathrm{W}(\;, u_i) = \mathrm{W}(\;, u_i) = \frac{q u_i - \bar{q} \bar{u}_i}{q - \bar{q}}, \; \mathrm{W}(\;, u_i u_j) = \mathrm{W}(\;, u_i u_j) = 1,$$

$$\mathrm{W}(\;, u_i u_j) = \mathrm{W}(\;, u_i u_j) = \frac{q^2 u_i u_j - \bar{q}^2 \bar{u}_i \bar{u}_j}{q^4 - \bar{q}^4}, \quad \mathrm{W}(\;, u_i u_j) = \mathrm{W}(\;, u_i u_j) = \frac{q^2 \bar{u}_i \bar{u}_j - \bar{q}^2 u_i u_j}{q^4 - \bar{q}^4}.$$

- As a consequence, if $q^2 + \bar{q}^2 = 1$, then we get that

$$|\mathrm{OSASM}(2n+1)| = \left(Z_{2n+1}(\underbrace{1, \ldots, 1}_{2n+1})/s \right)\Big|_{s=0}.$$

- Then by identifying certain properties that uniquely characterizes the partition function, it is obtained that

$$Z_{2n+1}(u_1, u_2, \ldots, u_{2n+1}) = \prod_{1 \le i < j \le 2n+1} \frac{\sigma(q^2 u_i u_j)\sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \operatorname*{Pf}_{0 \le i < j \le 2n+1} \left(\begin{cases} s & i = 0, \\ \frac{\sigma(u_i \bar{u}_j) Z_2(u_i, u_j)}{\sigma(q^2 u_i u_j)\sigma(q^2 \bar{u}_i \bar{u}_j)} & i \ge 1 \end{cases} \right),$$

where

$$\sigma(x) = \frac{x - \bar{x}}{q^4 - \bar{q}^4} \quad \text{and} \quad Z_2(u_i, u_j) = \frac{s^2 (q^2 \bar{u}_i \bar{u}_j - \bar{q}^2 u_i u_j)}{(q^4 - \bar{q}^4)} + \frac{(q u_i - \bar{q} \bar{u}_i)(q u_j - \bar{q} \bar{u}_j)}{(q - \bar{q})^2}.$$

- If $q^2 + \bar{q}^2 = 1$, then

$$\left(Z_{2n+1}(u_1, \ldots, u_n, \bar{u}_1, \ldots, \bar{u}_n, 1)/s \right)\Big|_{s=0} = 3^{-n^2} \prod_{i=1}^{n} \frac{(u_i + \bar{u}_i)^2}{u_i^2 + \bar{u}_i^2 - 1} \mathrm{sp}_{(n,n,\ldots,0,0)}(u_1^2, \ldots, u_n^2, \bar{u}_1^2, \ldots, \bar{u}_n^2, 1, -1),$$

where sp_λ is the irreducible symplectic character of $\mathrm{Sp}_{2n+2}(\mathbb{C})$.

- Lastly, evaluating the symplectic character at $(\underbrace{1, \ldots, 1}_{2n+1}, -1)$ completes the proof.

A symmetry property for even-order OSASMs

- For any ASM A, let $\mathrm{R}(A)$ be the number of nonzero entries in the strictly upper triangular part of A and $\mathrm{T}(A)$ be the column index of the 1 in the first row of A.

Theorem (Kumari, 2026)

For indeterminates r and t

$$\sum_{A \in \mathrm{OSASM}(2n)} r^{\mathrm{R}(A)} t^{\mathrm{T}(A)} = \sum_{A \in \mathrm{OSASM}(2n)} r^{\mathrm{R}(A)} t^{2n+2-\mathrm{T}(A)}.$$

- Equivalently,

$$\left| \{ M \in \mathrm{OSASM}(2n) : \mathrm{R}(M) = r_0, \ \mathrm{T}(M) = t_0 \} \right| = \left| \{ M \in \mathrm{OSASM}(2n) : \mathrm{R}(M) = r_0, \ \mathrm{T}(M) = 2n+2-t_0 \} \right|,$$

for any r_0 and t_0.

Acknowledgements

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Reference

[1] N. Kumari, Off-diagonally symmetric alternating sign matrices.
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