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Rectangle partitions generalizing integer partitions

Krystian Gajdzica[a] Robin Visser[b] Maciej Zakarczemny[c]

[a]Theoretical Computer Science Department, Jagiellonian University, Poland
[b]Department of Algebra, Charles University, Czech Republic
[c]Department of Applied Mathematics, Cracow University of Technology, Poland

Variations of the partition function $p(n)$

- **The A-partition function $p_A(n)$:** the number of sequences $(\lambda_1, \lambda_2, \dots, \lambda_r)$ satisfying

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_r \quad \text{and} \quad \lambda_1, \lambda_2, \dots, \lambda_r \in A,$$

where A is a fixed **multiset of positive integers**
- **A-partitions are considered the same** if they differ only in the order of their parts

Examples

- $A = \mathbb{N}$: the partition function
- $A = \{m^i : i \in \mathbb{N}_0\}$: the m-ary partition function
- $A = \{\underbrace{1, 1, \dots, 1}_{k}, \underbrace{2, 2, \dots, 2}_{k}, \dots\}$: the k-colored partition function
- $A = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots\}$: the plane partition function

Tiling of a rectangle

- **The rectangular partition function $p(m, n)$:** the number of tilings of the $m \times n$ rectangle into rectangular blocks with integer sides
- **Partitions of a rectangle are considered the same** if they consist of the same multiset of blocks (their geometric arrangement does not matter)

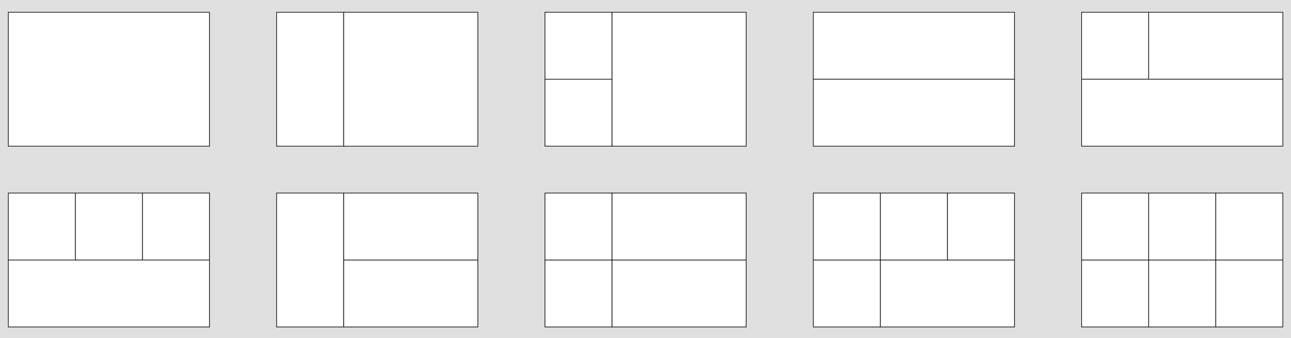

Figure: All rectangular partitions corresponding to $p(2, 3) = 10$.

Basic properties of $p(2, n)$

- $p(1, n) = p(n)$
- $p(n) + \sum_{i=1}^{n} p(i) \leqslant p(2, n) \leqslant \sum_{i=0}^{n} \binom{p(n-i)+1}{2} p(i)$
- $p(2, n) \leqslant p(n)^2 + \sum_{i=1}^{n} \left(p(i) + p(i-1)\right) p(n-i)^2$
- The function $p(2, n)$ satisfies **Benford's Law**: for every string of digits f in base $b \geqslant 2$ we have

$$\lim_{x \to \infty} \frac{|\{1 \leqslant n \leqslant x : p(2, n) \text{ in base } b \text{ begins with } f\}|}{x} = \log_b(f+1) - \log_b(f)$$

The Hardy–Ramanujan theorem for rectangular partitions

- $p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}$
- $p(2, n) \sim \frac{\pi \sqrt[4]{2}}{32 n^{7/4}} e^{\pi\sqrt{2n}}$

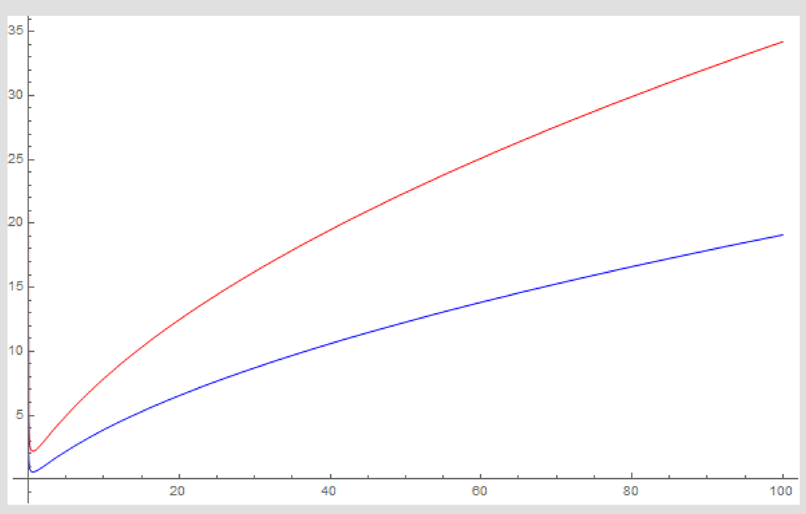

Figure: $\ln(p(n))$ & $\ln(p(2, n))$ for $n \leqslant 100$.

Ingredients of the proof:

- The asymptotic of non-unitary partitions "$p(n) - p(n-1)$"
- Erdős-Nicolas-Sárközy theorem on partitions of n without a given subsum
- Hardy-Ramanujan asymptotic for $p(n)$
- Murty theorem on asymptotic of the convolution of two power series

Restricted partitions of a rectangle

- **The restricted rectangular partition function $p_{k,\ell}(2, n)$:** the number of partitions of the rectangle $2 \times n$ into blocks $1 \times 1, 1 \times 2, \dots, 1 \times k$ and $2 \times 2, 2 \times 3, \dots, 2 \times \ell$

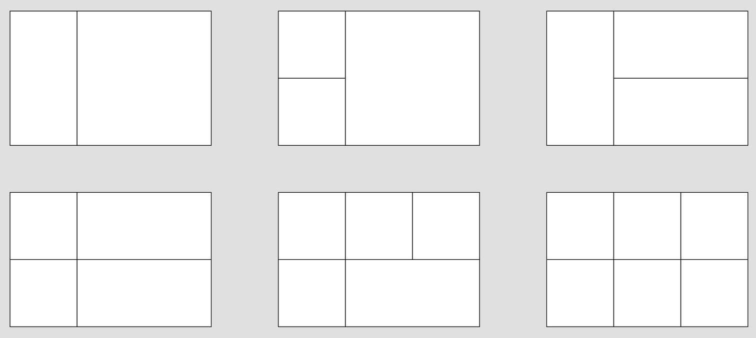

Figure: All restricted rectangular partitions corresponding to $p_{2,2}(2, 3) = 6$.

Fundamental properties of $p_{k,\ell}(2, n)$

Initial formulae for $p_{k,\ell}(2, n)$

$k \setminus \ell$	1	2	3
1	1	$\frac{n}{2} + A_{1,2}(n)$	$\frac{n^2}{12} + \frac{n}{2} + A_{1,3}(n)$
2	$n+1$	$\frac{n^2}{4} + n + A_{2,2}(n)$	$\frac{n^3}{36} + \frac{7n^2}{24} + \frac{11n}{12} + A_{2,3}(n)$
3	$\frac{n^2}{3} + n + A_{3,1}(n)$	$\frac{n^3}{18} + \frac{5n^2}{12} + \frac{5n}{6} + A_{3,2}(n)$	$\frac{n^4}{216} + \frac{2n^3}{27} + \frac{7n^2}{18} + A_{3,3}(n)$

Table: The formulae for $p_{k,\ell}(2, n)$ for $1 \leqslant k, \ell \leqslant 3$, where $A_{i,j}(n)$ are quasi-polynomial parts.

- The first column is essential: $p_{k,\ell}(2, n) = p_{k,\ell-1}(2, n) + p_{k,\ell}(2, n-\ell)$
- The first row is related to A-partition functions: $p_{1,\ell}(2, n) = p_{\{1,2,\dots,\ell\}}(n)$
- The entries could suggest that $p_{k,\ell}(2, n)$ is a **quasi-polynomial**
- The problem shows up for $p_{4,1}(2, n)$, because of

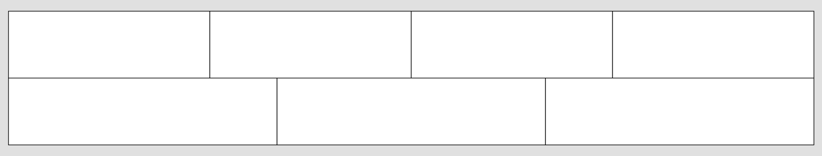

Figure: A partition of the rectangle 2×12 into 4 blocks 1×3 and 3 blocks 1×4.

Conjecture: For $k \geqslant 1$, the function $p_{k,1}(2, n)$ is a quasi-polynomial in n for all large n, with degree $k - 1$ and with a quasi-period dividing $\operatorname{lcm}(1, 2, \dots, k)$.

Theorem: For $k, \ell \geqslant 1$, we have $p_{k,\ell}(2, n) = \frac{2^{k-1} n^{k+\ell-2}}{\ell! k! (k+\ell-2)!} + O(n^{k+\ell-3})$.

Restricted m-ary partitions of a rectangle

- **The restricted rectangular m-ary partition function $b_{i,j}(2, n)$:** the number of partitions of the rectangle $2 \times n$ into blocks $1 \times 1, 1 \times m, \dots, 1 \times m^i$ and $2 \times m, 2 \times m^2, \dots, 2 \times m^j$

Theorem: For $m \geqslant 3$, $i, j \geqslant 0$ and $n = \sum_{t=0}^{k} c_t m^t$, we have

$$b_{i,j}(2, n) \equiv \prod_{t=1}^{i} (2c_t + 1) \prod_{t=1}^{j} (c_t + 1) \pmod{m}.$$

If $m = 2$, $i, j \geqslant 0$ and $i \neq 0$, we have

$$b_{i,j}(2, n) \equiv \prod_{t=0}^{j} (c_t + 1) \pmod{m}.$$

Questions, questions, questions...

- Is there an analogue of the Hardy-Ramanujan theorem for $p(m, n)$?
- What about other variations of the rectangular partition function?
- Are there intriguing divisibility properties of $p(m, n)$, $p_{k,\ell}(m, n)$, $b_{i,j}(m, n)$,...?
- Is the sequence $(p(m, n))_{n=1}^{\infty}$ log-concave?

References

[1] K. Gajdzica, R. Visser, and M. Zakarczemny, *"Rectangle Partitions Generalizing Integer Partitions"*, Ann. Comb. (2026). The research was supported by the National Science Center grant no. 2024/53/N/ST1/01538.