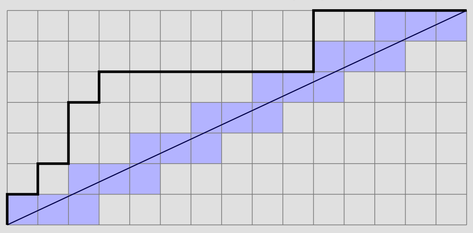
Lattice Path Conference
20-24 July 2026
TU Wien
Vienna, Austria

Symmetric statistics on rational Dyck paths

Lilan Dai Nankai University
Shishuo Fu Chongqing University
Dun Qiu Nankai University

Rational Dyck paths

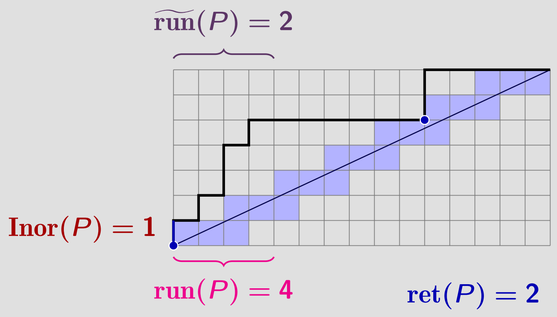
- $\mathcal{D}_{m,n}$ is the set of paths from $(0,0)$ to (m,n) with steps E, N, staying above $y = \frac{n}{m}x$.
- For a composition $\alpha \vDash n$, let
$$\mathcal{D}_{m,\alpha} := \{P \in \mathcal{D}_{m,|\alpha|} : \mathrm{comp}(P) = \alpha\}.$$
- Coarea sequence: $\mathrm{U}(P) = (u_1, \ldots, u_n)$, where u_i counts cells left of the ith north step.



$P \in \mathcal{D}_{15,(1,1,2,1,2)}, \quad \mathrm{U}(P) = (0,1,2,2,3,10,10).$

Four statistics

- **Return**: $\mathrm{ret}(P)$ is the number of N steps on the boundary of diagonal cells.
- **Run**:
$$\mathrm{run}(P) = \min\{\, i \in [n] : i \notin \{u_1, \ldots, u_n\}\,\}.$$
It counts peaks before the first EE.
- **Ratio-run**: for $k = \lfloor m/n \rfloor$,
$$\widetilde{\mathrm{run}}(P) = \min\{\, i \in [n] : ki \notin \{u_1, \ldots, u_n\}\,\}.$$
- **Initial north**: $\mathrm{Inor}(P)$ is the number of initial consecutive N steps.



Signatures and initial overlap

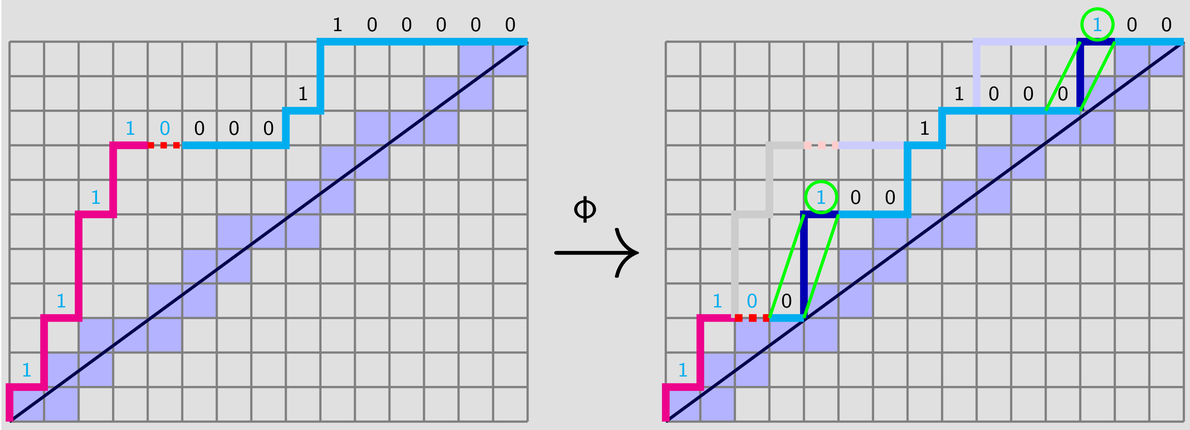
- An east step has signature 1 if it follows a north step, and 0 otherwise. For example,
$$N\,E\,E\,N\,E \quad \Longrightarrow \quad \mathbf{s} = 101.$$
- Let $r = \mathrm{rr}(\alpha, m)$, resp. $r = \widetilde{\mathrm{rr}}(\alpha, m)$, be the largest integer i for which the first i vertical components contribute to both $(\mathrm{run}, \mathrm{ret})$, resp. $(\widetilde{\mathrm{run}}, \mathrm{ret})$.

Main result I: Run–return symmetry

Let $m \geq n$, $\alpha \vDash n$, and $r = \mathrm{rr}(\alpha, m)$. Let
$$\mathcal{D}^{(k,\ell)}_{m,\alpha} := \{P \in \mathcal{D}_{m,\alpha} : \mathrm{run}(P) = k,\ \mathrm{ret}(P) = \ell\}.$$
Then, for all non-negative integers $a \leq b$,
$$\left|\mathcal{D}^{(r+a,r+b-a)}_{m,\alpha}\right| = \left|\mathcal{D}^{(r+b,r)}_{m,\alpha}\right|.$$
This identity implies
$$\sum_{\mathcal{D}_{m,\alpha}} p^{\mathrm{run}(P)} q^{\mathrm{ret}(P)} = \sum_{\mathcal{D}_{m,\alpha}} p^{\mathrm{ret}(P)} q^{\mathrm{run}(P)}.$$

Run–return example

Here $m = 15$, $\alpha = (1,2,3,2,1,2)$, and the map sends a path with $(\mathrm{run}, \mathrm{ret}) = (4,2)$ to one with $(\mathrm{run}, \mathrm{ret}) = (2,4)$ while keeping the same plain signature.



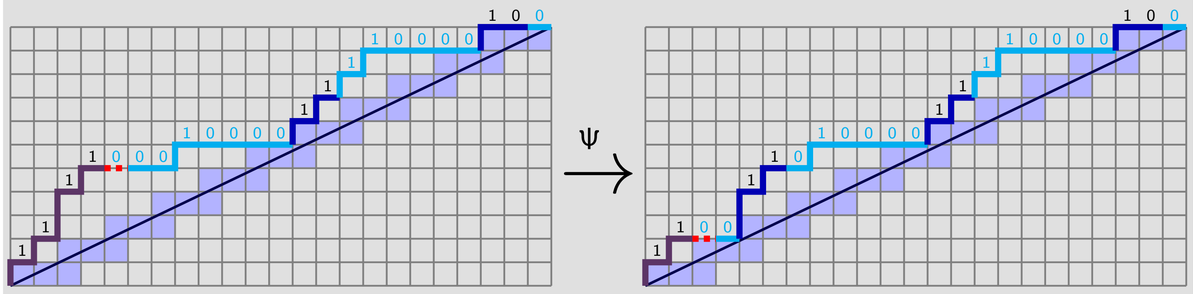
$$P \in \mathcal{D}^{(4,2)}_{15,(1,2,3,2,1,2)} \quad \longmapsto \quad \Phi(P) \in \mathcal{D}^{(2,4)}_{15,(1,2,3,2,1,2)}.$$

Main result II: Ratio-run–return symmetry

Let $m \geq n$, $\alpha = (\alpha_1, \ldots, \alpha_\ell) \vDash n$, and $r = \widetilde{\mathrm{rr}}(\alpha, m)$. Let
$$\widetilde{\mathcal{D}}^{(a,b)}_{m,\alpha} := \{P \in \mathcal{D}_{m,\alpha} : \widetilde{\mathrm{run}}(P) = a,\ \mathrm{ret}(P) = b\}.$$
Then, for all non-negative integers $a \leq b$,
$$\left|\widetilde{\mathcal{D}}^{(r+a,r+b-a)}_{m,\alpha}\right| = \left|\widetilde{\mathcal{D}}^{(r+b,r)}_{m,\alpha}\right|.$$
This identity implies
$$\sum_{\mathcal{D}_{m,\alpha}} p^{\widetilde{\mathrm{run}}(P)} q^{\mathrm{ret}(P)} = \sum_{\mathcal{D}_{m,\alpha}} p^{\mathrm{ret}(P)} q^{\widetilde{\mathrm{run}}(P)}.$$

Ratio-run–return example

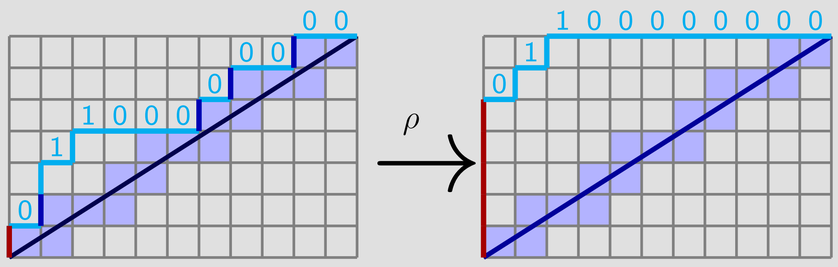
For $m = 23$, $k = \lfloor 23/11 \rfloor = 2$. The map sends $(\widetilde{\mathrm{run}}, \mathrm{ret}) = (2,3)$ to $(1,4)$ and preserves ratio-signatures.



$$P \in \widetilde{\mathcal{D}}^{(2,3)}_{23,(1,1,2,1,1,1,1,1,1,1)} \quad \longmapsto \quad \Psi(P) \in \widetilde{\mathcal{D}}^{(1,4)}_{23,(1,1,2,1,1,1,1,1,1,1)}.$$

Main result III: Return–initial-north symmetry

Let
$$D^{(i,j)}_{m,n} := \{P \in \mathcal{D}_{m,n} : \mathrm{ret}(P) = i,\ \mathrm{Inor}(P) = j\}.$$
Then, for all non-negative integers a, b,
$$\left|D^{(a,b)}_{m,n}\right| = \left|D^{(b,a)}_{m,n}\right|.$$



$$P \in D^{(5,1)}_{11,7} \quad \longmapsto \quad \rho(P) \in D^{(1,5)}_{11,7}.$$

Since Inor is the first part of $\mathrm{comp}(P)$, $\Phi\rho\Phi$ swaps run with Inor, while $\Psi\rho\Psi$ swaps $\widetilde{\mathrm{run}}$ with Inor when $m \geq n$.

Symmetric joint distributions

For any m, n, we have
$$\sum_{\mathcal{D}_{m,n}} p^{\mathrm{ret}(P)} q^{\mathrm{Inor}(P)} = \sum_{\mathcal{D}_{m,n}} p^{\mathrm{Inor}(P)} q^{\mathrm{ret}(P)}.$$
In particular, for $m \geq n$, we have
$$\sum_{\mathcal{D}_{m,n}} p^{\mathrm{run}(P)} q^{\mathrm{Inor}(P)} = \sum_{\mathcal{D}_{m,n}} p^{\mathrm{Inor}(P)} q^{\mathrm{run}(P)},$$
$$\sum_{\mathcal{D}_{m,n}} p^{\widetilde{\mathrm{run}}(P)} q^{\mathrm{Inor}(P)} = \sum_{\mathcal{D}_{m,n}} p^{\mathrm{Inor}(P)} q^{\widetilde{\mathrm{run}}(P)}.$$

Classical and Fuss–Catalan generating functions

For classical Dyck paths,
$$C(t,p,q) = \sum_{n \geq 0} t^n \sum_{P \in \mathcal{D}_{n,n}} p^{\mathrm{run}(P)} q^{\mathrm{ret}(P)}$$
has the closed form
$$C(t,p,q) = \frac{2pq(1-t) - 2(p+q) + 4 + 2(p+q-pq-pqt)\sqrt{1-4t}}{(1-pqt)(2-p+p\sqrt{1-4t})(2-q+q\sqrt{1-4t})}.$$
For k-Dyck paths,
$$C^{(k)}(t,p,q,r) = \sum_{n \geq 0} t^n \sum_{P \in \mathcal{D}_{kn,n}} p^{\mathrm{run}(P)} q^{\mathrm{ret}(P)} r^{\mathrm{nc}(P)},$$
$C^{(k)}$ can be written in terms of $C^{(k)}_r = C^{(k)}(t,1,1,r)$:
$$C^{(k)}(t,p,q,r) = \frac{C^{(k)}_r\left((p-1)(q-1)C^{(k)}_r + (p+q) - pq\right)}{\left((p-1)C^{(k)}_r - p\right)\left((q-1)C^{(k)}_r - q\right)},$$
where $\mathrm{nc}(P)$ denotes the number of vertical components of P.

Conclusions

- Two rational extensions of the run statistic, run and $\widetilde{\mathrm{run}}$, both have symmetric joint distributions with ret on $\mathcal{D}_{m,n}$ for $m \geq n$.
- The involutions Φ and Ψ exchange the two statistics while preserving composition type.
- An involution ρ exchanges ret and Inor; together with Φ and Ψ gives symmetries for $(\mathrm{run}, \mathrm{Inor})$ and $(\widetilde{\mathrm{run}}, \mathrm{Inor})$ when $m \geq n$.
- These results extend classical Dyck-path symmetries and raise new questions for rational paths with $m < n$.

References

[1] S. Elizalde, M. Rubey, *Symmetries of statistics on lattice paths between two boundaries*, Adv. Math., 287 (2016): 347-388.
[2] Y. Li and Z. Lin, *A symmetry on parking functions via Dyck paths*, Discrete Math. 346 (2023), no. 6, Paper No. 113426, 9 pp.
[3] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.