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Partial Degenerate Stirling Numbers

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Stirling numbers of the second kind - generalisations

classical	$\left\{ {n \atop k} \right\}$	$\frac{(e^x-1)^k}{k!}$	$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k-1} \right\} + k \left\{ {n \atop k} \right\}$	$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k$	# of partitions of $[n]$ into k non-empty blocks
generalized	$\left\{ {n \atop k} \right\}_{\alpha,\beta,\gamma}$	$\frac{(e_\alpha^\beta(x)-1)^k}{\beta^k k!} e_\alpha^\gamma(x)$	$\left\{ {n+1 \atop k} \right\}_{\alpha,\beta,\gamma} = \left\{ {n \atop k-1} \right\}_{\alpha,\beta,\gamma} + (k\beta - n\alpha + \gamma) \left\{ {n \atop k} \right\}_{\alpha,\beta,\gamma}$	$(x)_{n;\alpha} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\alpha,\beta,\gamma} (x-\gamma)_{k;\beta}$	# of (α,β,γ)−partitions of $[n]$ into k+1 non-empty sets

Restricted/associate Stirling numbers

restricted	$\left\{ {n \atop k} \right\}_{\leq \ell}$	$\frac{(e_{\leq \ell}-1)^k}{k!}$	each block has at most ℓ elements
associate	$\left\{ {n \atop k} \right\}_{\geq \ell}$	$\frac{(e^x - e_{<\ell})^k}{k!}$	each block has at least ℓ elements

(α,β,γ)-partitions

- $\alpha,\beta,\gamma,n,k \in \mathbb{N}$ s.t. $\alpha|\gamma$, $\alpha|\beta$, $k \leq n$.
- cells have cyclic ordered compartments; order: # of compartments
- α-closing property: each element is placed at a time, and for each available α compartments the first compartment may be occupied by an element

(α,β,γ)-**partition** is a partition of $[n]$ into $k+1$ cells with α-closing properties s.t.

- a possible empty cell of order γ
- k non-empty cells of order β with a ,,favorite" compartment

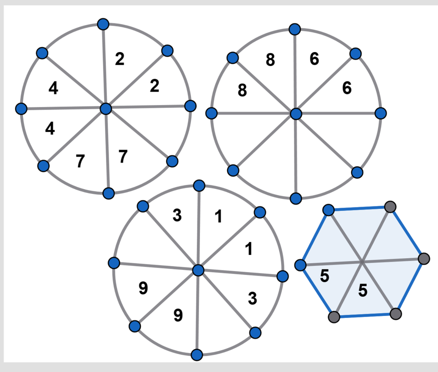

Figure: (2,8,6)-partition

Weighted mixed partitions

- $\mathcal{A}$ ring, $\alpha,\beta,\gamma \in \mathcal{A}$; $n,k \in \mathbb{N}$
- G: possible empty subset of $[n]$
- $P_k = B_1/B_2/\cdots/B_k$: set partition of $[n] \setminus G$ into non-empty blocks
- $\Pi^*_{n,k}$: distributions of $[n]$ into a pair (G,P_k)

$$\left\{ {n \atop k} \right\}_{\alpha,\beta,\gamma} = \sum_{(G,P_k)\in \Pi^*_{n,k}} w((G,P_k)),$$

- $w(G) = (\gamma)_{|G|;\alpha}$
- $w(B_i) = (\beta-\alpha)_{|B_i|-1;\alpha} \to w(P_k) = \prod_{i=1}^{k} B_i$
- $w((G,P_k)) = w(G)w(P_k)$

$$w(\{1,3,9\}\{2,4,7\}\{6,8\}|\{5\}) = \gamma(\beta-\alpha)^3(\beta-2\alpha)^2$$

Degenerate restricted Stirling numbers

D1 # of (α,β,γ)-partitions of $[n]$ in $k+1$ cells s.t. the cells of order β contain at most ℓ elements.

D2

$$\left\{ {n \atop k} \right\}^{\leq \ell}_{\alpha,\beta,\gamma} = \sum_{(G,P_k)\in \Pi^{\leq \ell}_{n,k}} w((G,P_k)),$$

$\Pi^{\leq \ell}_{n,k}$: all distributions of $[n]$ into a pair (G,P_k) s.t. $|B_i| \leq \ell$

Generating function:

$$\sum_{n=0}^{\infty} \left\{ {n \atop k} \right\}^{\leq \ell}_{\alpha,\beta,\gamma} \frac{x^n}{n!} = e_\alpha^\gamma(x) \frac{\left(e^\beta_{\alpha;\leq \ell}(x)-1\right)^k}{\beta^k k!}.$$

Recursive relation:

$$\left\{ {n+1 \atop k} \right\}^{\leq \ell}_{\alpha,\beta,\gamma} = \gamma \left\{ {n \atop k} \right\}^{\leq \ell}_{\alpha,\beta,\gamma-\alpha} + \sum_{i=k-1}^{n-\ell-1} \binom{n}{i} (\beta-\alpha)_{n-i,\alpha} \left\{ {i \atop k-1} \right\}^{\leq \ell}_{\alpha,\beta,\gamma}.$$

Generalized associated Stirling numbers (a special case)

D1 $(0,1,\gamma)$-partitions s.t. the cells of order 1 contain more than ℓ elements.

D2

$$\left\{ {n \atop k} \right\}^{>\ell}_{\gamma} = \sum_{(G,P_k)\in \Pi^{>\ell}_{n,k}} \gamma^{|G|}.$$

$\Pi^{>\ell}_{n,k}$: all distributions of $[n]$ into a pair (G,P_k) s.t. $B_i > \ell$.

Generating function:

$$\sum_{n=kl}^{\infty} \left\{ {n \atop k} \right\}^{>\ell}_{\gamma} \frac{x^n}{n!} = \frac{e^{\gamma x}\left(e^x - e_{\leq \ell}(x)\right)^k}{k!}.$$

Recursive relation:

$$\left\{ {n+1 \atop k} \right\}^{>\ell}_{\gamma} = \gamma \left\{ {n \atop k} \right\}^{>\ell}_{\gamma} + \sum_{i=0}^{n} \gamma^i \binom{n}{i} \left\{ {n-i \atop k} \right\}_{\geq \ell+1}.$$

Partial degenerate Stirling numbers - $S^{\gamma,\alpha,\beta}_{n,k,\ell}$

D1
- $n,k,\ell,\alpha,\beta,\gamma \in \mathbb{N}$ s.t. $\alpha|\gamma$, $\alpha|\beta$, $k \leq n$, $\ell \leq n$.
- free cell: compartments may contain more elements ($\alpha = 0$)

$S^{\gamma,\alpha,\beta}_{n,k,\ell}$: # of distribution of $[n]$ into $k+1$ cells s.t.

1. a free cell of order γ may be empty
2. free cells of order 1 contain more than ℓ elements - subsets
3. cells of order β with α-closing property and favorite compartment contain at most ℓ elements.

D2

$$S^{\gamma,\alpha,\beta}_{n,k,\ell} = \sum_{(G,P_k)\in \Pi^*_{n,k}} w((G,P_k)),$$

- $w(G) = \gamma^{|G|}$
- $w(B_i) = 1$, if $|B_i| > \ell$
- $w(B_i) = (\beta-\alpha)_{|B_i|-1,\alpha}$ if $|B_i| \leq \ell$

$$\ell = 2 \Rightarrow w(\{1,3,9\}\{2,4,7\}\{6,8\}|\{5\}) = \gamma(\beta-\alpha)$$

Generating function:

$$\sum_{n=0}^{\infty} S^{\gamma,\alpha,\beta}_{n,k,\ell} \frac{x^n}{n!} = \frac{e^{\gamma x}}{k!} \left(e^x + \sum_{i=1}^{\ell} \frac{(\beta|\alpha)_i}{\beta} \frac{x^i}{i!} - \sum_{i=0}^{\ell} \frac{x^i}{i!} \right)^k.$$

Connection:

$$S^{\gamma,\alpha,\beta}_{n,k,\ell} = \sum_{i=0}^{n} \sum_{j=0}^{k} \binom{n}{i} \left\{ {i \atop j} \right\}^{>\ell}_{\gamma} \left\{ {n-i \atop k-j} \right\}^{\leq \ell}_{\alpha,\beta}.$$

Open question

Can we define a ,,nice" lattice path model for

- generalised Stirling numbers
- partial degenerate Stirling numbers

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